Yu.L. KLIMONTOVICH

KINETIC THEORY OF NONIDEAL GASES AND NONIDEAL PLASMAS

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Preface

There are many books on kinetic theory of gases and plasmas on the market. What was the motivation of the author in writing another book on this subject ?

The foundations of present-day non-equilibrium statistical theory of gases and plasmas is due to the important work of N.N. Bogolyubov, M. Born, A.A. Vlasov, H. Green, J. Kirkwood, J. Yvon, L.D. Landau and I.R. Prigogine. In this book we attempt to present some of the ideas and methods of their work. They will be applied to a complete description of the kinetic processes in nonideal gases and plasmas and to the derivation of the kinetic theory of long-range fluctuations. The latter is important, in particular, for the description of turbulent processes, defining the so-called anomalous transport processes.

Clearly, the simplest case is the one of weak nonideality, in which it is possible to introduce a small parameter: the density parameter or the plasma parameter. For denser systems one uses model equations, as in equilibrium theory. In this field, the developments are at present still very preliminary.

The theory is constructed as a generalization of the kinetic theory of ideal gases and plasmas. It is therefore important to analyze the limitations of the usual kinetic equations. The book is written with great detail; therefore it should be of use not only to research physicists, but also to professors, and to graduate students of various specializations.

The book consists of three parts. The first part is devoted to the classical kinetic theory of nonideal gases, the second to the classical kinetic theory of fully ionized plasmas, and the third to the quantum kinetic theory of nonideal gases and plasmas. The concluding chapter presents a short account of the kinetic theory of chemically reacting systems and of partially ionized plasmas. This chapter was included in order to indicate some directions of further generalizations of the present results. and to attract attention upon this important and interesting problem.

The main stress is laid here on the fundamental aspects of the theory. Relatively little space is given to the applications. Whenever possible, the reader is directed towards additional literature.

In several places of this book I used results obtained in collaboration with my students and collaborators: V.V. Belyi, YU. A. Kukharenko, W. Ebeling, W. Kraeft, V.A. Puchkov, E.F. Slin'ko. The collaboration with them was for me not only useful, but also pleasant.

K.P. Gurov was the first person who read my work on the kinetic theory of nonideal gases and plasmas. He also read the manuscript of this book. I am very grateful to him for his help. I also gratefully acknowledge the remarks and discussions about the manuscript with V.V. Belyi, L.M. Gorbunov, M.E. Marinchuk and A.A. Rukhadze.

The Author

PART I

Kinetic Theory of Nonideal Gases

INTRODUCTION

The basis of the kinetic theory consists of the equations for the one-particle distribution function: the kinetic equations. Typical kinetic equations are those of Boltzmann, of Vlasov, of Landau, and of Balescu and Lenard.

All the kinetic equations are approximate; therefore they provide a simplified description of the statistical processes in gases and plasmas. There are phenomena, which cannot be described in terms of the known kinetic equations. In order to include such phenomena in the description, the assumptions made in the derivations of these equations must be weakened and these equations must be generalized.

Thus, in deriving the Boltzmann equation from the Liouville equation (or from the corresponding BBGKY hierarchy) one makes use of the smallness of the density parameter ε. Similarly, for a plasma one assumes that the plasma parameter μ is small.

The parameters ε and μ characterize the role of the interactions in the kinetic equations for gases or plasmas. This role is twofold. On the one hand, it defines the relaxation processes responsible, for instance, for the approach to equilibrium. In other words, the interaction determine the dissipative processes in gases and plasmas.

On the other hand, the interactions contribute to the non-dissipative properties, e.g., the thermodynamic functions (internal energy, pressure, entropy, etc.). These contributions of the interactions are responsible for the deviations of these quantities from their ideal value.

In the Boltzmann, Landau or Balescu-Lenard equations, the interactions determine only the dissipative characteristics. In this sense, these equations can be

called kinetic equations for the ideal gas or plasma. One of the problems of the present book is the construction of kinetic equations for nonideal gases and plasmas. Within their corresponding models — the binary collision approximation or the polarization approximation — these equations take account of the contributions of the interactions to both the dissipative and the non-dissipative properties [17, 18]. (see also refs [6, 67, 68, 71, 73]).

The second problem of the book is the construction of kinetic equations for dense gases. The first difficulty in this direction is the derivation of equations taking into account both binary and triple collisions. Such an equation was derived by Choh and Uhlenbeck [5], by using Bogolyubov's expression for the two-particle distribution function to the first order in the density parameter. This kinetic equation is not quite consistent. In the dissipative characteristics it takes account of both binary and triple collisions, but in the non-dissipative properties it retains only binary collisions. In this book we derive an equation in which the triple collisions are treated more completely.

In ref. [4], Bogolyubov developed a method by which, assuming the complete weakening of the initial correlations, he expands the two-body correlations systematically in powers of the density parameter. Clearly, this method also leads to an expansion of the collision integral of the kinetic equation in powers of the density. However, the realization of Bogolyubov's programme faces some difficulties of principle. The investigations of Weinstock [19], Goldman and Freeman [20], Dorfman and Cohen [21], showed that the collision integral, including four-body and higher order collisions, diverges [22].

The solution of these difficulties leads to the modification of the basic assumptions underlying the kinetic equations. It was shown that the complete weakening of the initial correlations must be replaced by the more flexible assumption of the partial weakening of these correlations [23, 24]. By using this assumption, we derive from the Liouville equation an equation for the smoothed distribution function in phase space. From the latter we derive a hierarchy for the smoothed distribution functions. It differs from the BBGKY hierarchy in retaining explicitly the dissipation due to binary collisions. If this hierarchy is solved by assuming the complete weakening of the initial correlations in a time shorter than the binary collision relaxation time, the Boltzmann equation is recovered.

For denser gases, we obtain from the smoothed hierarchy a kinetic equation whose collision integral is convergent. In this way, one may construct kinetic equations taking into account four-body, five-body collisions, etc. But these equations become more and more complicated. One therefore uses a more convenient method, analogous to the one used in equilibrium statistical mechanics of dense

gases and fluids. Instead of a kinetic equation for the one-particle distribution, one rather uses a set of equations for the one-particle distribution and for the binary correlations.

From the hierarchy for the smoothed distribution functions we may derive kinetic equations taking into account the long-range fluctuations. From the latter we may derive hydrodynamic equations in which the viscosity and the thermal conductivity are determined not only by the collisions, but also by the long-range fluctuations.

In the derivation of the Boltzmann equation, one assumes implicitly the continuity of the collision process defining the collision integral. This amounts to describing the distribution function as a deterministic (non-fluctuating) quantity. Taking into account the discreteness of the collision processes leads to fluctuations of the distribution function. These fluctuations have a range much longer than the one of the fluctuations defining the collision integral. In order to describe the former, we may consider the Boltzmann equation as a Langevin equation with a given source of fluctuations: the latter was first studied by Kadomtsev [25]. The development of the kinetic theory of the equilibrium and non-equilibrium fluctuations in gases is another problem of our book (chapter 4). The corresponding theory for plasmas is studied in chapter 11.

CHAPTER 1

The Method of Distribution Functions and the Method of Moments

1. EQUATIONS FOR THE POSITION AND MOMENTUM DISTRIBUTION FUNCTIONS IN A GAS OF MONATOMIC PARTICLES

The microscopic mechanical state of a monatomic particle gas at time t is defined by the specification of the positions $\boldsymbol{r}_1, \ldots, \boldsymbol{r}_N$ and of momenta $\boldsymbol{p}_1, \ldots, \boldsymbol{p}_N$ of all the N particles. For conciseness, we introduce the notations: $x_i \equiv (\boldsymbol{r}_i, \boldsymbol{p}_i)$, a six-dimensional vector defining the state of the particle labelled i $(1 \leqslant i \leqslant N)$, and $x \equiv (x_1, \ldots, x_N)$, a $6N$-dimensional vector defining the state of the complete system.

The distribution function of the variables x is denoted by $f_N(x, t)$. The expression $f_N(x, t)\,dx$ represents the probability that, at time t, the coordinates and momenta of the particles have values within a range dx around x. The function f_N is normalized as follows: $\int dx\, f_N(x, t) = 1$.

Let $\Phi(|\boldsymbol{r}_i - \boldsymbol{r}_j|) \equiv \Phi_{ij}$ denote the potential energy of central interaction of the pair of particles i, j. Let also m denote the mass of the atoms. Then the Hamiltonian H of the gas can be written as:

$$H = \sum_{1 \leqslant i \leqslant N} \left(\frac{\boldsymbol{p}_i^2}{2m} + u(\boldsymbol{r}_i)\right) + \sum_{1 \leqslant i < j \leqslant N} \Phi_{ij} \tag{1.1}$$

where $u(\boldsymbol{r}_i)$ is the potential energy of an atom in the external field.

The distribution function $f_N(x, t)$ obeys the Liouville equation:

$$\frac{\partial f_N}{\partial t} + \sum_{1 \leqslant i \leqslant N} \left(\frac{\partial H}{\partial \boldsymbol{p}_i} \cdot \frac{\partial f_N}{\partial \boldsymbol{r}_i} - \frac{\partial H}{\partial \boldsymbol{r}_i} \cdot \frac{\partial f_N}{\partial \boldsymbol{p}_i}\right) = 0 \tag{1.2}$$

We now use Hamilton's equations for the particles:

$$\dot{\boldsymbol{r}}_i = \frac{\partial H}{\partial \boldsymbol{p}_i} \equiv \frac{\boldsymbol{p}_i}{m} \equiv \boldsymbol{v}_i \tag{1.3}$$

$$\dot{\boldsymbol{p}}_i = -\frac{\partial H}{\partial \boldsymbol{r}_i} = -\sum_{\substack{1 \leqslant j \leqslant N \\ j \neq i}} \frac{\partial \Phi_{ij}}{\partial \boldsymbol{r}_i} + \boldsymbol{F}_0(\boldsymbol{r}_i, t) \tag{1.3}$$

Equation (1.2) can then be rewritten

$$\frac{\partial f_N}{\partial t} + \sum_{1 \leqslant i \leqslant j} \boldsymbol{v}_i \cdot \frac{\partial f_N}{\partial \boldsymbol{r}_i} + \boldsymbol{F}(\boldsymbol{r}_i) \cdot \frac{\partial f_N}{\partial \boldsymbol{p}_i} = 0 \tag{1.4}$$

We introduced here the expression $\boldsymbol{F}(\boldsymbol{r}_i, t)$ representing the force acting on the particle labelled i :

$$\boldsymbol{F}(\boldsymbol{r}_i, t) = -\frac{\partial}{\partial \boldsymbol{r}_i} \sum_{\substack{1 < j < N \\ j \neq i}} \Phi_{ij} + \boldsymbol{F}_0(\boldsymbol{r}_i, t) \tag{1.5}$$

where $\boldsymbol{F}_0 = -\partial u / \partial \boldsymbol{r}_i$ is the external force.

We now consider several different forms of the Liouville equation. We introduce the following operator notations:

$$\hat{\theta}_{ij} = \frac{\partial \Phi_{ij}}{\partial \boldsymbol{r}_i} \cdot \frac{\partial}{\partial \boldsymbol{p}_i} + \frac{\partial \Phi_{ij}}{\partial \boldsymbol{r}_j} \cdot \frac{\partial}{\partial \boldsymbol{p}_j} \tag{1.6}$$

$$\hat{H}_{x_1, \ldots, x_s} = \sum_{1 \leqslant i \leqslant s} \left(\boldsymbol{v}_i \cdot \frac{\partial}{\partial \boldsymbol{r}_i} + \boldsymbol{F}_0 \cdot \frac{\partial}{\partial \boldsymbol{p}_i} \right) - \sum_{1 \leqslant i < j \leqslant s} \hat{\theta}_{ij} \tag{1.7}$$

$$\hat{L}^{(0)}_{x_1, \ldots, x_s} = \frac{\partial}{\partial t} + \sum_{1 \leqslant i \leqslant s} \left(\boldsymbol{v}_i \cdot \frac{\partial}{\partial \boldsymbol{r}_i} + \boldsymbol{F}_0 \cdot \frac{\partial}{\partial \boldsymbol{p}_i} \right) \tag{1.8}$$

Using the definitions (1.6), (1.7), the Liouville equation can be written as:

$$\left\{ \frac{\partial}{\partial t} + \sum_{1 \leqslant i \leqslant N} \left(\boldsymbol{v}_1 \cdot \frac{\partial}{\partial \boldsymbol{r}_i} + \boldsymbol{F}_0 \cdot \frac{\partial}{\partial \boldsymbol{p}_i} \right) - \sum_{1 \leqslant i < j \leqslant N} \hat{\theta}_{ij} \right\} f_N = 0 \tag{1.9}$$

or

$$\left\{ \frac{\partial}{\partial t} + \hat{H}_{x_1, \ldots, x_N} \right\} f_N = 0 \tag{1.10}$$

Finally, using the operator (1.8), Eq. (1.9) can also be written as:

$$\left\{ \hat{L}^{(0)}_{x_1, \ldots, x_s} - \sum_{1 \leqslant i < j \leqslant N} \hat{\theta}_{ij} \right\} f_N = 0 . \tag{1.11}$$

The Liouville equation is a linear partial differential equation of first order: its characteristic equations are just Hamilton's equations (1.3). It follows that the solution of the Liouville equation requires the solution of the equations of motion of all the particles in the system. This problem can generally not be solved for the large systems considered here.

Thus, the Liouville equation cannot be solved in general form. However, in practice, the general solution is not necessary. For the description of the processes in a gas it is usually sufficient to know the distribution functions of one and of two particles, viz., $f_1(x_1, t)$ and $f_2(x_1, x_2, t)$. These are related in the following way to the function $f_N(x, t)$:

$$f_1(x_1, t) = V \int dx_2 \ldots dx_N \; f_N(x_1, \ldots, x_N, t) \tag{1.12}$$

$$f_2(x_1, x_2, t) = V^2 \int dx_3 \ldots dx_N \; f_N(x_1, \ldots, x_N, t) \tag{1.13}$$

where V is the volume of the system. From these definitions follows the normalization property of these functions:

$$\frac{1}{V} \int dx_1 \, f_1 = 1 \; , \quad \frac{1}{V^2} \int dx_1 \, dx_2 \, f_2 = 1 \tag{1.14}$$

The distribution function of a set of s particles can be defined in a similar way:

$$f_s = V^s \int dx_{s+1} \ldots dx_N \; f_N(x_1, \ldots, x_N, t) \; ; \; \frac{1}{V^s} \int dx_1 \ldots dx_s \, f_s = 1 \tag{1.15}$$

From the definitions (1.12) and (1.13) follows that $V^{-1} f_1 \, dx_1$ represents the probability that the position and momentum of the first particle lies within limits $d\boldsymbol{r}_1 \, d\boldsymbol{p}_1$ around $\boldsymbol{r}_1, \boldsymbol{p}_1$, whatever the positions and momenta of the other particles in the gas. A similar interpretation holds for $V^{-2} f_2 \, dx_1 \, dx_2$ and for the higher distribution functions.

We now show that the most important functions needed in a gas-dynamic description can be determined from the sole knowledge of the distribution function f_1.

In gas dynamics, the most important properties are the number density of the particles, the momentum (or velocity) density and the kinetic energy density (or temperature). In a non-stationary state, these functions depend on the position and on the time:

$$n = n(\boldsymbol{r}, t) \; , \quad \boldsymbol{u} = \boldsymbol{u}(\boldsymbol{r}, t) \; , \quad W = W(\boldsymbol{r}, t)$$

where W is the kinetic energy density.

The quantities $n, n\boldsymbol{u}, W$ are average values of the corresponding microscopic quantities $n^{\mathbf{M}}$, $(n\boldsymbol{u})^{\mathbf{M}}$, $W^{\mathbf{M}}$ which are defined as follows

$$n^{\mathbf{M}}(\boldsymbol{r}, t) = \sum_{1 \leqslant i \leqslant N} \delta\left(\boldsymbol{r} - \boldsymbol{r}(t)\right)$$

$$n^{\mathbf{M}} \boldsymbol{u}^{\mathbf{M}}(\boldsymbol{r}, t) = \sum_{1 \leqslant i \leqslant N} \boldsymbol{v}_i \, \delta\left(\boldsymbol{r} - \boldsymbol{r}_i(t)\right)$$

$$W^{\mathbf{M}}(\boldsymbol{r}, t) = \sum_{1 \leqslant i \leqslant N} (\boldsymbol{p}_i^2 / 2\boldsymbol{m}) \, \delta\left(\boldsymbol{r} - \boldsymbol{r}_i(t)\right) \tag{1.16}$$

Using the definition

$$\langle A \rangle = \int dx\; A(x)\, f_N(x\,,\,t)$$

and taking account of the properties of the δ-function:

$$\int dx'\; \delta(x-x') = 1\,, \quad \int dx'\; \delta(x-x')\, f(x') = f(x)$$

we obtain

$$n(\boldsymbol{r}\,,t) = \int dx \sum_{1\leqslant i\leqslant N} \delta(\boldsymbol{r}-\boldsymbol{r}_i(t))\, f_N = N\int dx\; \delta(\boldsymbol{r}-\boldsymbol{r}_1(t))\; f_N$$

$$= \frac{N}{V}\int d\boldsymbol{p}_1\; d\boldsymbol{r}_1\; \delta(\boldsymbol{r}-\boldsymbol{r}_1)\, f_1(\boldsymbol{r}_1\,,\boldsymbol{p}_1,t) = \frac{N}{V}\int d\boldsymbol{p}\, f_1(\boldsymbol{r}\,,\boldsymbol{p},t) \qquad (1.17)$$

and similarly

$$n(\boldsymbol{r}\,,t)\,\boldsymbol{u}(\boldsymbol{r}\,,\,t) = (N/V)\int d\boldsymbol{p}\;(\boldsymbol{p}/m)\; f_1(\boldsymbol{r},\boldsymbol{p},t)$$

$$W(\boldsymbol{r}\;,\,t) = (N/V)\int d\boldsymbol{p}\;(p^2/2m)\, f_1(\boldsymbol{r},\boldsymbol{p},t) \qquad (1.18)$$

From Eqs (1.17), (1.18) follows that for the statistical description of the gas-dynamical variables, the knowledge of the one-particle distribution function $f_1(\boldsymbol{r},\boldsymbol{p},t)$ is sufficient. The factor N/V entering Eqs (1.17), (1.18) represents the number of particles per unit volume, averaged over the whole volume. This quantity will be denoted by n: $N/V=n$. The quantity $1/n$ represents the specific volume, i.e. the volume per particle.

As the definition of many important averages requires only the knowledge of the one-particle distribution function, we are justified in making an effort for deriving equations which would only involve this function f_1 .

In order to obtain such an equation, we multiply Eq. (1.4) for f_N by V and integrate it over all variables except the first. We consider the result of this operation for each term separately.

$$V\int dx_2 \ldots dx_N \frac{\partial f_N}{\partial t} = \frac{\partial f_1}{\partial t} \quad .$$

Here we used Eq. (1.12) defining f_1. In order to integrate the second term we must taken into account the fact that the distribution function tends to zero at the boundary of the volume enclosing the system. We thus obtain:

$$V\int dx_2 \ldots dx_N \sum_{1\leqslant i\leqslant N} \boldsymbol{v}_i \cdot \frac{\partial f_N}{\partial \boldsymbol{r}_i} = \boldsymbol{v}_1 \cdot \frac{\partial f_1}{\partial \boldsymbol{r}_1} \quad .$$

For the integration of the third term. we take into account the fact that the distribution function vanishes as the momenta tend to infinity; using also the definition (1.13) of f_2, we obtain :

$$V \int dx_2 \dots dx_N \sum_{1 \leq i \leq N} \frac{\partial}{\partial \boldsymbol{r}_i} \left(\sum_{\substack{1 \leq j \leq N \\ j \neq i}} \Phi_{ij} \right) \cdot \frac{\partial f_N}{\partial \boldsymbol{p}_i}$$

$$= V(N-1) \int dx_2 \dots dx_N \frac{\partial \Phi_{12}}{\partial \boldsymbol{r}_1} \cdot \frac{\partial f_N}{\partial \boldsymbol{p}_1} = \frac{N-1}{V} \int dx_2 \frac{\partial \Phi_{12}}{\partial \boldsymbol{r}_1} \cdot \frac{\partial f_2}{\partial \boldsymbol{p}_1} .$$

Using the fact that $N \gg 1$, and the notations (1.8) and $n = N/V$, we find the following equation for the one-particle distribution function:

$$\left\{ \frac{\partial}{\partial t} + \boldsymbol{v}_1 \cdot \frac{\partial}{\partial \boldsymbol{r}_1} + \boldsymbol{F}_0 \cdot \frac{\partial}{\partial \boldsymbol{p}_1} \right\} f_1$$

$$\equiv \hat{L}^{(0)}_{\mathbf{x}_1} f_1 = n \int dx_2 \frac{\partial \Phi_{12}}{\partial \boldsymbol{r}_1} \cdot \frac{\partial f_2(x_1, x_2, t)}{\partial \boldsymbol{p}_1} . \tag{1.19}$$

We see that this equation is not closed for f_1, as it also involves the two-particle distribution function. Similarly, integrating Eq. (1.4) over all the variables except those of two particles, we find, for $N \gg 1$, the equation for the function f_2:

$$\left(\hat{L}^{(0)}_{x_1, x_2} - \hat{\theta}_{12} \right) f_2 = n \int dx_3 \left(\frac{\partial \Phi_{13}}{\partial \boldsymbol{r}_1} \cdot \frac{\partial}{\partial \boldsymbol{p}_1} + \frac{\partial \Phi_{23}}{\partial \boldsymbol{r}_2} \cdot \frac{\partial}{\partial \boldsymbol{p}_2} \right) f_3$$

$$\equiv n \int dx_3 \; (\hat{\theta}_{13} + \hat{\theta}_{23}) f_3 . \tag{1.20}$$

Here we used the definitions (1.6), (1.8).

Thus the equation for f_2 is not closed, as it involves the distribution function f_3. The equation for f_3 involves f_4, and so on. We thus obtain a hierarchical chain of equations. For the determination of the function f_1 we need to know all the distribution functions of the system, up to f_N. This chain of equations is called the Bogolyubov hierarchy, or the BBGKY hierarchy (Bogolyubov-Born-Green-Kirkwood-Yvon). The general equation for f_s $(2 \leq s \leq N)$ can be written as follows:

$$\hat{L}^{(0)}_{x_1, \dots, x_s} f_s - \sum_{1 \leq i < j \leq s} \hat{\theta}_{ij} f_s = n \sum_{1 \leq i \leq s} \int dx_{s+1} \hat{\theta}_{i, s+1} f_{s+1} . \tag{1.21}$$

Because of the complication of this hierarchy, we must try to set up approximation schemes by which we can determine closed equations for the simplest of these distribution functions. An example of such an equation is the well-known Boltzmann equation, which is a closed equation for the one-particle distribution function $f_1(x_1, t)$.

The price to be paid in the transition from the Liouville equation (or from

the BBGKY hierarchy) to the simpler closed equations is in a coarser description of the processes in the system. For the investigation of a possible transition to an approximate description, it is necessary to discuss the relations among characteristic time scales and spatial parameters of the monatomic gas.†

2. THE APPROXIMATION OF BINARY COLLISIONS

We first consider the simplest model of a gas, in which the atoms are idealized as hard spheres of diameter r_0. The potential energy of a pair of atoms is then given by the following function:

$$\Phi(r) = \begin{cases} 0 & \text{for} \quad r > r_0 \\ \infty & \text{for} \quad r \leqslant r_0 \end{cases} \tag{2.1}$$

Thus, the atoms do not interact when their relative distance is larger than r_0. The latter is a characteristic parameter of the system. Another characteristic length is the average interatomic distance: $r_{\mathbf{av}} \sim (1/n)^{\frac{1}{3}}$.

From these two parameters we may construct a dimensionless number:

$$\varepsilon = n r_0^3 \sim (r_0/r_{\mathbf{av}})^3 \tag{2.2}$$

ε will be called the density parameter.

For atmospheric pressure and room temperature, the density n is $n \approx 10^{19}$ cm^{-3}. For monatomic gases, $r_0 \approx 2-3 \cdot 10^{-8}$ cm. For these values, we obtain $\varepsilon \sim 10^{-5} \ll 1$. The mean free path, which determines the transport coefficients of the gas, is expressed as follows in terms of r_0, $r_{\mathbf{av}}$:

$$l = \frac{1}{\sqrt{2}\,\pi r_0^2 n} \sim \frac{1}{n r_0^2} = \frac{r_0}{\varepsilon} .$$

We thus obtain the following relations among the three parameters:

$$r_0 : r_{\mathbf{av}} : l \sim 1 : \varepsilon^{-\frac{1}{3}} : \varepsilon^{-1} .$$

Hence:

$$r_0 \ll r_{\mathbf{av}} \ll l \quad \text{for} \quad \varepsilon \ll 1 . \tag{2.3}$$

If the density parameter $\varepsilon \ll 1$ the gas will be called 'dilute'. The ideal gas is a limiting case of the dilute gas. Whenever $\varepsilon \lessapprox 1$, the gas will be called 'dense'.

For dilute gases, we may limit our considerations to the first approximation in ε. We must however take account of the following remarks.

The interatomic interactions play two roles in a gas. In the first place,

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† **For a discussion of this and the following sections, see also the recent books of Balescu [67] and of Résibois and De Leener [68]. (Transl.)**

they contribute to the thermodynamic properties of the gas: internal energy, entropy, pressure, etc. In the ideal gas limit, the contribution of the interactions to the thermodynamic properties is zero. In the dilute gas, there exist additional terms in the thermodynamic functions, as compared to those of the ideal gas. These terms are proportional to ε. More generally, the contributions of the interactions to the thermodynamic functions can be represented as series in ε.

The interactions also define the relaxation properties of the gas, for instance, the process of approach to equilibrium and the various transport processes, such as viscosity, heat conduction, diffusion, etc.

The characteristic time of approach to equilibrium (or relaxation time) is of the order of magnitude of the mean free time between successive collisions: $t_{\mathbf{rel}} = l/v_T$, where v_T is the average thermal speed. To zeroth order in the density parameter ε the mean free path is infinite, because $l \sim r_0/\varepsilon$. Hence, the relaxation time is also infinite, i.e., there is no relaxation at all. This result is natural, because the relaxation process is determined by the interactions.

The transport processes cannot be defined either in zeroth approximation in ε, because the transition to a hydrodynamic description requires the condition $l \ll L$, where L is, for instance, the size of the system. As the mean free path is infinitely long in this approximation, the condition cannot be satisfied.

Thus, in order to obtain kinetic equations, i.e. closed equations for the function f_1, describing relaxation processes, it is necessary to consider at least terms of first order in the parameter ε. In this approximation one also obtains the contributions of the interactions to the thermodynamic properties.

In the language of interparticle collisions, the first approximation in ε corresponds to considering only binary collisions. This means that the contribution of processes in which more than two particles approach each other at a distance of order r_0 is negligibly small.

We now show that, in the binary collision approximation, the infinite hierarchy of equations for the functions $f_1, f_2, \ldots$ can be truncated after the second one; we thus obtain a closed set of two equations for f_1 and f_2. For this purpose, it is useful to introduce the relations between the distribution functions $f_1, f_2, \ldots$ and the corresponding correlation functions $g_2, g_3, \ldots$

By definition, we have:

$$f_2(x_1, x_2, t) = f_1(x_1, t) f_1(x_2 t) + g_2(x_1, x_2, t) \qquad (2.4)$$

For $g_2 = 0$, $f_2 = f_1 f_1$ and the particles 1 and 2 are statistically independent.

The three-particle correlation function $g_3(x_1, x_2, x_3, t)$ is defined by the equation:

$$f_3(1,2,3) = f_1(1)f_1(2)f_1(3) + f_1(1)g_2(2,3) + f_1(2)g_2(1,3) + f_1(3)g_2(1,2) + g_3(1,2,3) . \quad (2.5)$$

For simplicity, we denoted here the set of variables x_j of particle j by the single letter j.

By removing any of the three particles to an infinite distance with respect to the other two the function g_3 vanishes. Consider for instance the case where particle 3 is removed to infinity. Then $g_3 = 0$, and also $g_2(1,3) = g_2(2,3) = 0$ and thus using (2.4):†

$$f_3(1,2,3) = f_1(1)f_1(2)f_1(3) + f_1(3)g_2(1,2) = f_2(1,2)f_1(3).$$

From the definition (1.15) of $f_1, f_2, \ldots$ in terms of f_N, we conclude that f_s can be expressed in terms of f_{s+1} as follows:

$$f_s = V^{-1}\int dx_{s+1}\, f_{s+1} . \quad (2.6)$$

In particular

$$f_1 = V^{-1}\int dx_2\, f_2 , \quad f_2 = V^{-1}\int dx_3\, f_3 . \quad (2.7)$$

Substituting Eq. (2.4) into the first equation (2.7) and using the normalization (1.14) of f_1, we obtain:

$$V^{-1}\int dx_1\, g_2(1,2) = V^{-1}\int dx_2\, g_2(1,2) = 0 . \quad (2.8)$$

Similarly, from (2.5), (2.7) and (2.8) we find

$$V^{-1}\int dx_1\, g_3(1,2,3) = V^{-1}\int dx_2\, g_3(1,2,3) = V^{-1}\int dx_3\, g_3(1,2,3) = 0 \quad (2.9)$$

These properties of the correlation functions g_2, g_3 will be used below.

In the binary collision approximation, the configurations in which three particles are at distance r_0 are not considered. Therefore, g_3 can be neglected in Eq. (2.5).

We now substitute (2.4) into the right hand side of Eq. (1.19) and introduce the following notation for the average force acting on particle 1:

$$\boldsymbol{F}(\boldsymbol{r}_1, t) = \boldsymbol{F}_0(\boldsymbol{r}_1, t) - n\frac{\partial}{\partial \boldsymbol{r}_1}\int dx_2\, \Phi_{12} f_1(x_2, t) . \quad (2.10)$$

Here the first term represents the external force, and the second term represents the average force due to the surrounding particles, whose spatial density distribution is

$$n(\boldsymbol{r}_2, t) = n\int d\boldsymbol{p}_2\, f_1(x_2, t) .$$

† **The correlations are thoroughly discussed in Yvon's book [69]. See also [67]. (Transl.)**

We rewrite the equation for f_1 as follows

$$\left(\frac{\partial}{\partial t}+\boldsymbol{v}_1\cdot\frac{\partial}{\partial \boldsymbol{r}_1}+\boldsymbol{F}\cdot\frac{\partial}{\partial \boldsymbol{p}_1}\right)f_1\equiv\hat{L}_{\mathbf{x}_1}\, f_1=I(x_1,t) \tag{2.11}$$

The operator $\hat{L}_{x_1\ldots x_s}$ differs from $\hat{L}^{(0)}_{x_1\ldots x_s}$ by the substitution of $\boldsymbol{F}$ for $\boldsymbol{F}_0$ [see eq. (1.8)].

In Eq. (2.11) we introduced the notation

$$I(x_1,t)=n\int dx_2\,\frac{\partial\Phi_{12}}{\partial \boldsymbol{r}_1}\cdot\frac{\partial\, g_2(x_1,x_2,t)}{\partial \boldsymbol{p}_1}\equiv n\int dx_2\,\hat{\theta}_{12}\,g_2\,. \tag{2.12}$$

This term represents the rate of change of the distribution function as a result of correlations between particles 1 and 2. In chapter 2 it will be shown that in the hard sphere model the correlations differ from zero only when the centres of the particles are at a distance r_0, i.e. when the particles are colliding. Therefore, the expression (2.12) will be called the 'collision integral'.

For $F_0=0$ and for spatially homogeneous systems, in which $f_1=f_1(\boldsymbol{p}_1,t)$ and $g_2=g_2(\boldsymbol{r}_1-\boldsymbol{r}_2,\ \boldsymbol{p}_1,\boldsymbol{p}_2,t)$, Eq. (2.11) simplifies:

$$\frac{\partial f_1}{\partial t}=n\int\, d\boldsymbol{r}_2\,d\boldsymbol{p}_2\,\frac{\partial\Phi_{12}}{\partial \boldsymbol{r}_1}\cdot\frac{\partial}{\partial \boldsymbol{p}_1}\,g_2(\boldsymbol{r}_1-\boldsymbol{r}_2,\ \boldsymbol{p}_1,\boldsymbol{p}_2,t) \tag{2.13}$$

We now transform Eq. (1.20) for f_2. We substitute into the right-hand side expression (2.5) with $g_3=0$. We consider the result of this substitution in the first term of the r.h.s., containing the function Φ_{13}:

$$n\int dx_3\frac{\partial\Phi_{13}}{\partial \boldsymbol{r}_1}\cdot\frac{\partial}{\partial \boldsymbol{p}_1}\Big\{f_1(1)\,f_1(2)\,f_1(3)+f_1(1)\,g_2(2,3)$$
$$+f_1(2)\,g_2(1,3)+f_1(3)\,g_2(1,2)\Big\}. \tag{2.14}$$

Using Eq. (2.4), the first and last terms in the sum can be rewritten as

$$n\int dx_3\ \frac{\partial\Phi_{13}}{\partial \boldsymbol{r}_1}\,f_1(x_3,t)\cdot\frac{\partial}{\partial \boldsymbol{p}_1}\,f_2(x_1,x_2,t)\ .$$

We bring this term to the left-hand side of Eq. (1.20) and combine it with $\boldsymbol{F}_0(\partial f_2/\partial \boldsymbol{p}_1)$. Using the definition (2.10) of the average force, we obtain in the result the term

$$\boldsymbol{F}(\boldsymbol{r}_1,t)\cdot\frac{\partial f_2(1,2)}{\partial \boldsymbol{p}_1}\ . \tag{2.15}$$

The second term in (2,14) vanishes in the binary collision approximation, because it differs from zero only when the three particles are simultaneously near to each other. Indeed, the interaction Φ_{13} vanishes when particles 1 and 3 are

at a distance larger than r_0, and the correlation $g_2(2,3)$ vanishes whenever particles 2 and 3 are far apart.

Using the definition (2.12) of the collision integral, as well as Eq. (2.11), the penultimate term in Eq. (2.14) can be rewritten as:

$$I(x_1,t)\,f_1(x_2,t) = \hat{L}_{x_1} f_1(x_1,t)\,f_1(x_2,t)\ . \tag{2.16}$$

As a result of the substitution of (2.5) into the second term of the r.h.s. of (1.20), containing the function Φ_{23}, we obtain, respectively, instead of (2.15), (2.16):

$$\boldsymbol{F}(\boldsymbol{r}_2,t)\ \partial f_2/\partial \boldsymbol{p}_2\ ,\quad I(x_2,t)\,f_1(x_1,t) = \hat{L}_{x_2} f_1(x_2,t) f_1(x_1,t)\ . \tag{2.17}$$

We thus find for Eq. (1.20), in the binary collision approximation, the following form:

$$\hat{L}_{x_1 x_2} f_2 - \hat{\theta}_{12} f_2 = \hat{L}_{x_1 x_2} f_1(x_1,t)\ f_1(x_2,t)\ . \tag{2.18}$$

We recall that the operator $\hat{L}_{x_1,\ldots,x_s}$ is defined by Eq. (1.8), in which $\boldsymbol{F}_0$ is changed into $\boldsymbol{F}$.

Using Eq. (2.4), Eq. (2.18) can also be written in the form:

$$\hat{L}_{x_1 x_2} g_2(x_1,x_2,t) - \hat{\theta}_{12} f_2(x_1,x_2,t) = 0 \tag{2.19}$$

Equations (2.11) and (2.18) or (2.19) form a closed set for the distribution functions $f_1(x_1,t)$, $f_2(x_1,x_2,t)$ in the binary collision approximation.

We call attention to the structure of Eqs (2.18), (2.19). The left hand side of (2.18) represents the rate of change of the function f_2 as a result of the motion of the pair of interacting particles. The right hand side does not depend explicitly on f_2: it describes the change of the functions $f_1(x_1,t)$, $f_1(x_2,t)$ as a result of the motion of the non-interacting particles 1 and 2 under the action of the average field. Rewriting (2.18) in the form (2.19), we clearly see that the operator $\hat{L}_{x_1 x_2}$ acts only on the correlation function g_2, not on the complete function f_2.

For $\boldsymbol{F}_0 = 0$ and for a spatially homogeneous system, Eq. (2.18) takes the simpler form

$$\left\{\frac{\partial}{\partial t} + \boldsymbol{v}_1\cdot\frac{\partial}{\partial \boldsymbol{r}_1} + \boldsymbol{v}_2\cdot\frac{\partial}{\partial \boldsymbol{r}_2} - \frac{\partial\Phi_{12}}{\partial \boldsymbol{r}_1}\cdot\frac{\partial}{\partial \boldsymbol{p}_1} - \frac{\partial\Phi_{12}}{\partial \boldsymbol{r}_2}\cdot\frac{\partial}{\partial \boldsymbol{p}_2}\right\} f_2$$
$$= \frac{\partial}{\partial t} f_1(\boldsymbol{p}_1,t)\, f_1(\boldsymbol{p}_2,t)\ . \tag{2.20}$$

In chapter 2 we will see how Eqs (2.11), (2.18) can be further simplified in the framework of the binary collision approximation. They will be replaced by a single closed equation for f_1: the Boltzmann equation.

3. THE WEAK COUPLING APPROXIMATION. THE POLARIZATION APPROXIMATION

The binary collision approximation considered in the previous paragraph is less general than the approximation of binary correlations ($g_3 = 0$). Indeed, in obtaining Eq. (2.18) we did not only set $g_3 = 0$ in the expression (2.5) for f_2; we also neglected certain terms involving the binary correlations g_2. Thus, when (2.5) was substituted into the first term of the r.h.s. of (1.20), the term

$$n\int dx_3\, \frac{\partial \Phi_{13}}{\partial \boldsymbol{r}_1} \cdot \frac{\partial f_1(x_1,t)}{\partial \boldsymbol{p}_1}\, g_2(2,3) \tag{3.1}$$

was neglected in (2.14), and the similar term

$$n\int dx_3\, \frac{\partial \Phi_{23}}{\partial \boldsymbol{r}_2} \cdot \frac{\partial f_1(x_1,t)}{\partial \boldsymbol{p}_2}\, g_2(1,3) \tag{3.2}$$

was neglected in the second term on the r.h.s. of (1.20). These terms involve the correlations of the surrounding particles with one of the two distinguished particles

We will see later that these terms play a fundamental role in plasmas, i.e. in mixtures of gases of charged particles. The interactions of charged particles decrease with the distance as $1/r$, i.e. much slower than the interactions of neutral atoms. As a result, the effective radius of interaction of the particles in a plasma (the Debye radius, r_D) is much larger than the average interparticle distance: $r_\mathrm{D} \gg r_\mathrm{av}$, where the effective interaction radius is r_0.

Thus, in the binary correlation approximation, the equation for the two-particle distribution function f_2 is:

$$\begin{aligned}\left(\hat{L}_{x_1 x_2} - \hat{\theta}_{12}\right) f_2(1,2) - \hat{L}_{x_1 x_2} f_1(1) f_1(2) &= \\ = n\int dx_3 \left\{\hat{\theta}_{13}\, g_2(2,3) f_1(1) + \hat{\theta}_{23} g_2(1,3)\, f_1(2)\right\} .\end{aligned} \tag{3.3}$$

This equation differs from Eq. (2.19) — pertaining to the binary collision approximation — by the presence of the right-hand side.

Equations (2.11), (3.3) also form a closed set for the distribution functions f_1, f_2, but now in the binary correlation approximation.

Later we shall consider also other approximate equations for f_2, besides the binary collision approximation. All these models can be obtained from (3.3) by making additional assumptions.

One of these approximations is related to the theory of perturbations with respect to the interaction strength of the particles. The dimensionless parameter in this case is the ratio of the potential energy (at a certain effective range of interparticle distances) to the average kinetic energy. Let us call this parameter η and consider the equation for f_2 to first order in η.

For $\eta \ll 1$, $g_2(1,2)$ is also much smaller than one, because the correlation

function is proportional to the interaction energy Φ_{12}. As a result, the right-hand side of Eq. (3.3) can be neglected, because it is of order η^2. In the left hand side, $f_2 = f_1 f_1 + g_2$ can be replaced by $f_1 f_1$ in the terms involving $\hat{\theta}_{12} f_2$. Thus, to first order in the interaction parameter, the 'weak coupling approximation', we get:

$$\hat{L}_{x_1 x_2} g_2 = \hat{\theta}_{12} f_1 f_1 \ . \tag{3.4}$$

The operator $\hat{\theta}_{12}$ appearing here was defined in Eq. (1.6).

This equation can also be obtained from Eqs. (2.18) or (2.19). Thus, the equation in the first-order approximation in the interaction parameter can be obtained both from the equation in the pair correlation approximation and from the more particular equation in the binary collision approximation.

We also consider another approximation, which will be called the polarization approximation. The justification of this name lies in the fact that this approximation is quite effective in a plasma, in which it describes the effect of the polarization of the medium.

The equation for the function g_2 in the polarization approximation is obtained from (3.3) by means of the following additional assumption. On the left-hand side, the function $f_2 = f_1 f_1 + g_2$ can be replaced by $f_1 f_1$ in the terms involving $\hat{\theta}_{12} f_2$ (as in the weak interaction approximation). This means that in the pair interaction process, the effect of the correlations is small, i.e. $g_2 \ll f_1 f_1$. But the contribution of the terms on the right-hand side of (3.3) can be large, even when $g_2 \ll f_1 f_1$, whenever each one of the particles 1 and 2 interacts simultaneously with many other particles. Such a situation is characteristic for plasmas.

Thus, in the polarization approximation, the equation for g_2 is:

$$\hat{L}_{x_1 x_2} g_2(1,2) = \hat{\theta}_{12} f_1(1) f_1(2) + n \int dx_3 \left\{ \hat{\theta}_{13} g_2(2,3)\, f_1(1) + \hat{\theta}_{23} g_2(1,3)\, f_1(2) \right\} . \tag{3.5}$$

This equation differs from Eq. (3.4) by the presence of terms describing the influence of surrounding particles on the distinguished pair. These terms will be called polarization terms.

4. THE EQUATION FOR THE PHASE SPACE DENSITY IN THE SIX-DIMENSIONAL POSITION AND MOMENTUM SPACE

The microscopic state of the gas is defined by the specification, at each time, of the set of positions and momenta of all the particles $x_1(t), \ldots, x_N(t)$. The function f_N describes the distribution averaged over all the microstates of the system.

For the description of the processes in a plasma, and also for the

description of the long range fluctuations in gases, a different description of the microstate is convenient. The microstate is defined when the density $N(x,t)$ is known at all times and in all points $x = (\boldsymbol{r}, \boldsymbol{p})$ of the six-dimensional phase space. This function is defined as follows [8,18] :

$$N(x,t) = \sum_{1 \leqslant i \leqslant N} \delta[x - x_i(t)] \ . \tag{4.1}$$

The function $N(x,t)$ is a microscopic quantity, as it depends on the values of the coordinates and of the momenta of the particles. The quantity

$$N(x,t)\,dx = \sum_{1 \leqslant i \leqslant N} \delta[x - x_i(t)]\,dx \tag{4.2}$$

represents the exact (not the average) number of particles in the volume element dx of the six-dimensional phase space, surrounding the point x. It actually is a random quantity, as it depends on all the $x_i(t)$.

All the microscopic quantities of the gas can be expressed in terms of the function $N(x,t)$, provided they are symmetric in the positions and momenta of all the particles. For instance, the Hamiltonian (1.1) can be represented as:

$$H = \int dx (p^2/2m)\, N(x\,,\,t) + \tfrac{1}{2}\int dx\, dx'\ \Phi(|\boldsymbol{r} - \boldsymbol{r}'|) N(x,t)\ N(x',t)\,. \tag{4.3}$$

The Hamiltonian (4.3) differs from (1.1) only by the fact that it includes the self-interaction of the particles. Whenever necessary, the corresponding term can be easily subtracted from (4.3).

The microscopic functions (1.16) can also be expressed in terms of $N(x,t)$:

$$\begin{aligned} n^{\mathrm{M}}(\boldsymbol{r},t) &= \int d\boldsymbol{p}\ N(x,t) \\ (n\boldsymbol{u})^{\mathrm{M}}_{\boldsymbol{r},t} &= \int d\boldsymbol{p}\,(\boldsymbol{p}/m)\ N(x,t) \\ W^{\mathrm{M}}(\boldsymbol{r},t) &= \int d\boldsymbol{p}\,(p^2/2m)\ N(x,t) \end{aligned} \tag{4.4}$$

The microscopic force, $\boldsymbol{F}^{\mathrm{M}}$ due to all the particles as well as to an external field, has the following expression:

$$\boldsymbol{F}^{\mathrm{M}}(\boldsymbol{r},t) = -\int dx'\ \frac{\partial\Phi(\boldsymbol{r}-\boldsymbol{r}')}{\partial \boldsymbol{r}}\, N(x',t) + \boldsymbol{F}_0(\boldsymbol{r},t)\ . \tag{4.5}$$

This expression differs from (1.5) by the inclusion of the self-force.

The equation for the function $N(x,t)$ can be obtained by using the equations of motion:

$$\frac{\partial N}{\partial t} + \boldsymbol{v}\cdot\frac{\partial N}{\partial \boldsymbol{r}} + \boldsymbol{F}^{\mathrm{M}}(\boldsymbol{r},t)\cdot\frac{\partial N}{\partial \boldsymbol{p}} = 0\ . \tag{4.6}$$

Substituting (4.5) into (4.6), one obtains a different form:†

$$\left\{\frac{\partial}{\partial t}+\boldsymbol{v}\cdot\frac{\partial}{\partial \boldsymbol{r}}+\boldsymbol{F}_0(\boldsymbol{r},t)\cdot\frac{\partial}{\partial \boldsymbol{p}} - \int dx'\ \frac{\partial\Phi(\boldsymbol{r}-\boldsymbol{r}')}{\partial \boldsymbol{r}}\cdot\frac{\partial}{\partial \boldsymbol{p}}\ N(x',t)\right\}N(x,t)=0\,. \tag{4.7}$$

Equations (4.6), (4.7) can be taken as the starting point for the construction of a statistical theory.

We consider a consequence of Eq. (4.7) related to the conservation of energy. We multiply (4.7) by $\boldsymbol{p}^2/2m$ and integrate over x. The first term gives:

$$\frac{\partial}{\partial t}\int dx(p^2/2m)\ N(x,t)$$

which represents the change of kinetic energy. By integrating over the position, the second term gives zero. The last term, integrated by parts over $\boldsymbol{p}$ and $\boldsymbol{r}$ yields:

$$-\int dx\,dx'\ \Phi(|\boldsymbol{r}-\boldsymbol{r}'|)\ N(x',t)\ \boldsymbol{v}\cdot\frac{\partial N(x,t)}{\partial \boldsymbol{r}} = \frac{\partial}{\partial t}\frac{1}{2}\int dx\,dx'\ \Phi(|\boldsymbol{r}-\boldsymbol{r}'|)\ N(x,t)\ N(x',t)$$

In order to obtain the last expression, the factor $\boldsymbol{v}\cdot(\partial N/\partial \boldsymbol{r})$ was eliminated by using (4.7). Finally, the contribution of the penultimate term is:

$$\boldsymbol{F}_0\cdot\int dx\ \boldsymbol{v}\ \ N(x,t)\ \ .$$

As a result, collecting all the terms, we obtain:

$$\frac{\partial}{\partial t}\left\{\int dx(p^2/2m)N(x,t)+\frac{1}{2}\int dx\,dx'\ \Phi(|\boldsymbol{r}-\boldsymbol{r}'|)\ N(x,t)\,N(x',t)\right\} = \boldsymbol{F}_0\cdot\int dx\,\boldsymbol{v}\ N(x,t)\equiv n^{\mathrm{M}}(\boldsymbol{r},t)\ \boldsymbol{u}^{\mathrm{M}}(\boldsymbol{r},t)\ \boldsymbol{F}_0\,. \tag{4.8}$$

This equation gives the balance of the total energy of the system.

5. THE METHOD OF MOMENTS

We now study the relation between the moments of the microscopic phase density $N(x,t)$ and the distribution and correlation functions.

Using the definition (1.12) of the function f_1 we obtain

$$\langle N(x,t)\rangle = \int d^{\mathrm{N}}y_i \sum_{1\leqslant i\leqslant N}\delta(x-y_i(t))\,f_N(y_1,\dots,y_N) = \frac{N}{V}\int dy_1\ \delta(x-y_1(t))\ f_1(y_1)$$

† **Equations (4.6),(4.7) are the classical analog of the corresponding quantum density operator in the Heisenberg representation (see sect. 71).**

It then follows that

$$\langle N(x,t)\rangle = nf_1(x,t)\ , \quad n = N/V\ . \tag{5.1}$$

Hence, the average phase density differs from f_1 only by its normalization:

$$V^{-1}\int dx\ f_1(x,t) = 1\ , \quad \int dx\,\langle N(x,t)\rangle = N\ .$$

Similarly, the second moment of the phase density is expressed as:

$$\langle N(x,t)N(x',t)\rangle = \frac{N(N-1)}{V^2}\, f_2(x,x',t) + \delta(x-x')\,\frac{N}{V} f_1(x,t) \tag{5.2}$$

It involves the two distribution functions f_1 and f_2 .

The third moment is expressed in terms of the distribution functions f_1, f_2, f_3:

$$\begin{aligned}\langle N(x,t)N(x',t)N(x'',t)\rangle &= \frac{N(N-1)(N-2)}{V^3}\, f_3(x,x',x'',t)\\ &+ \frac{N(N-1)}{V^2}\Big[\delta(x-x')\, f_2(x',x'',t) + \delta(x'-x'')\, f_2(x'',x,t)\\ &+ \delta(x''-x)\, f_2(x,x',t)\Big] + \frac{N}{V}\,\delta(x-x')\,\delta(x'-x'')\, f_1(x'',t)\end{aligned} \tag{5.3}$$

We now consider the expressions of the corresponding central moments. We introduce the following notation for the fluctuation of the phase density:

$$\delta N = N(x,t) - \langle N(x,t)\rangle \tag{5.4}$$

Using the fact that $\langle \delta N\rangle = 0$, we find:

$$\langle \delta N\,\delta N\rangle_{x,x',t} = \langle N(x,t)N(x',t)\rangle - \langle N(x,t)\rangle\langle N(x',t)\rangle \tag{5.5}$$

This expression is simply the definition of the second central moment. We now express it in terms of the functions f_1 and g_2. We use Eqs (5.2) and (5.1), as well as the definition (2.4), with the result:

$$\begin{aligned}\langle \delta N\,\delta N\rangle_{x,x',t} &= \frac{N(N-1)}{V^2}\, g_2(x,x',t)\\ &+ \frac{N}{V}\left\{\delta(x-x')\, f_1(x,t) - \frac{1}{V}\, f_1(x,t)\, f_1(x',t)\right\}\end{aligned} \tag{5.6}$$

It follows from the normalization condition (1.14) of f_1 and from the property (2.8) of g_2, combined with (5.6), that

$$\int dx'\,\langle \delta N\,\delta N\rangle_{x,x',t} = \int dx\,\langle \delta N\,\delta N\rangle_{x,x',t} = 0 \tag{5.7}$$

These properties are consequences of the conservation of the number of particles.

In the limit: $N \to \infty$, $V \to \infty$, $N/V = n =$ const, Eqs (5.2), (5.6) go over into the simpler forms:

$$\langle N(x,t)\,N(x',t)\rangle = n^2 f_2(x,x',t) + \delta(x-x')\,n f_1(x,t)$$

$$\langle \delta N\,\delta N\rangle_{x,x',t} = n^2 g_2(x,x',t) + \delta(x-x')\,n f_1(x,t)\ . \tag{5.8}$$

The third central moment is defined as follows

$$\begin{aligned}\langle \delta N\,\delta N\,\delta N\rangle_{x,x',x'',t} &= \langle N(x,t)N(x',t)N(x'',t)\rangle \\ &- \langle N(x,t)\rangle\langle \delta N\,\delta N\rangle_{x',x'',t} - \langle N(x',t)\rangle\langle \delta N\,\delta N\rangle_{x'',x,t} \\ &- \langle N(x'',t)\rangle\langle \delta N\,\delta N\rangle_{x,x',t} - \langle N(x,t)\rangle\langle N(x',t)\rangle\langle N(x'',t)\rangle\end{aligned} \tag{5.9}$$

In the limit $N \to \infty$, $V \to \infty$, $n =$ const, we find

$$\begin{aligned}\langle \delta N\,\delta N\,\delta N\rangle_{x,x',x'',t} &= n^3 g_3(x,x',x'',t) \\ &+ n^2\Big[\delta(x-x')\,g_2(x',x'',t) + \delta(x'-x'')\,g_2(x'',x,t) \\ &+ \delta(x''-x)\,g_2(x,x',t)\Big] + n\,\delta(x-x')\,\delta(x'-x'')\,f_1(x'',t)\ .\end{aligned} \tag{5.10}$$

Thus, the third central moment is expressed in terms of the functions f_1, g_2, g_3.

In all the previous expressions of the moments, all the suffices below the averaging sign correspond to the same time. Such moments are called one-time moments. All the previously considered distribution functions are also one-time functions. We now introduce the simplest two-time distribution functions and two-time moments.

The one-particle two-time distribution function represents the probability of finding one particle successively at point x at time t and at point x' at time t'. It is defined as follows:

$$F_1(x,t,x',t') = V\int d^{\mathbf{N}}y\ \delta(x-y_1(t))\,\delta(x'-y_1(t'))\,f_N(y,t') \tag{5.11}$$

We note that, for $t = t'$ we have

$$F_1(x,t,x',t) = \delta(x-x')\,f_1(x,t) \tag{5.12}$$

Integrating over x, we find

$$\int dx\,F_1(x,t,x',t') = f_1(x',t')\ . \tag{5.13}$$

Finally, using Liouville's theorem: $dy = dy'$, from which follows that $f_N(y,t) = f_N(y',t')$ we find:

$$\int dx'\ F_1(x,t,x',t') = f_1(x,t) \tag{5.14}$$

The two-particle two-time distribution function represents the probability of finding particle 1 at point x at time t, and particle 2 at point x' at time t':

$$f_2(x,t;x',t') = V^2 \int d^N y\, \delta(x-y_1(t))\,\delta(x'-y_2(t'))\, f_N(y,t') . \qquad (5.15)$$

For $t=t'$ this function coincides with the one-time distribution function $f_2(x, x', t)$.

The two-time moment is defined by analogy with (5.2)

$$\langle N(x,t)\,N(x',t')\rangle = \frac{N(N-1)}{V^2} f_2(x,t;x',t') + \frac{N}{V} F_1(x,t,x',t') \qquad (5.16)$$

For $t=t'$ this expression reduces to (5.2).

The corresponding expression for the two-time central moment is:

$$\langle \delta N\,\delta N\rangle_{x,t,x',t'} = \frac{N(N-1)}{V^2} g_2(x,t,x',t') + \frac{N}{V}\left\{F_1(x,t,x',t') - \frac{1}{V} f_1(x,t)\, f_1(x',t')\right\} \qquad (5.17)$$

Here:

$$g_2(x,t;x',t') = f_2(x,t;x',t') - f_1(x,t)\, f_1(x',t') \qquad (5.18)$$

is the two-time two-particle correlation function. It has properties analogous to (2.8). Using these properties, as well as Eqs (5.13), (5.14) and (5.17) we find

$$\int dx\, \langle \delta N\,\delta N\rangle_{x,t,x',t'} = \int dx' \langle \delta N\,\delta N\rangle_{x,t,x',t'} . \qquad (5.19)$$

In the limit $N\to\infty$, $V\to\infty$, n = const. (5.17) reduces to

$$\langle \delta N\,\delta N\rangle_{x,t;x',t'} = n^2 g_2(x,t;x',t') + n\, F_1(x,t,x',t') \qquad (5.20)$$

For $t=t'$ Eqs. (5.17), (5.20) reduce, respectively, to (5.6), (5.8).

More complicated two-time moments and distribution functions can be defined in a similar way.

6. THE APPROXIMATIONS OF THE FIRST AND SECOND MOMENTS. THE POLARIZATION APPROXIMATION

We now take the ensemble average of each term in Eq. (4.6) for the phase density $N(x,t)$. We make use of the identity:

$$\langle \boldsymbol{F}^{\mathbf{M}} N(x,t)\rangle = \boldsymbol{F}\langle N(x,t)\rangle + \langle \delta \boldsymbol{F}\,\delta N\rangle_{\boldsymbol{r},x,t} \qquad (6.1)$$

where $\boldsymbol{F}$ is the average force and $\delta\boldsymbol{F} = \boldsymbol{F}^{\mathbf{M}} - \boldsymbol{F}$. Taking into account the relation between the functions $\langle N(x,t)\rangle$ and $f_1(x,t)$, we find the following equation for $f_1(x,t)$:

$$\left[\frac{\partial}{\partial t} + \boldsymbol{v}\cdot\frac{\partial}{\partial \boldsymbol{r}} + \boldsymbol{F}(\boldsymbol{r},t)\cdot\frac{\partial}{\partial \boldsymbol{p}}\right] f_1 = -\frac{1}{n}\frac{\partial}{\partial \boldsymbol{p}}\cdot\langle \delta \boldsymbol{F}\,\delta N\rangle \equiv I(x,t) \tag{6.2}$$

where

$$\boldsymbol{F}(\boldsymbol{r},t) = \boldsymbol{F}_0 - n\int dx'\; f_1(x',t)\,\frac{\partial\Phi(|\boldsymbol{r}-\boldsymbol{r}'|)}{\partial \boldsymbol{r}} \;.$$

This equation corresponds to Eq. (2.11) for the distribution function $f_1(x_1,t)$. The only difference is the form of the collision integral. In (2.11) the latter is expressed in terms of the two-particle correlation function, whereas in (6.2) appears the one-time correlation of the fluctuations of force and of phase density. It is easy to show that these two forms are equivalent. Indeed, from (4.5) it follows that

$$\delta \boldsymbol{F}(\boldsymbol{r},t) = -\int dx'\;\delta N(x',t)\;\frac{\partial\Phi(|\boldsymbol{r}-\boldsymbol{r}'|)}{\partial \boldsymbol{r}}\,. \tag{6.3}$$

Hence we find the following relation between the functions $\langle \delta \boldsymbol{F}\delta N\rangle$ and $\langle \delta N\,\delta N\rangle$:

$$\langle \delta \boldsymbol{F}\,\delta N\rangle_{\boldsymbol{r},x,t} = -\int dx'\,\langle \delta N\,\delta N\rangle_{x',x,t}\,\frac{\partial\Phi(|\boldsymbol{r}-\boldsymbol{r}'|)}{\partial \boldsymbol{r}}\,. \tag{6.4}$$

Substituting the relation (5.8), we find the relation between $\langle \delta \boldsymbol{F}\,\delta N\rangle$ and g_2 and we prove the equivalence of (6.2) and (2.11).

Equation (6.2) is not closed, because along with the function f_1 it involves the new function $\langle \delta \boldsymbol{F}\,\delta N\rangle$. From (6.4) we see that for its calculation we need to know the function $\langle \delta N\delta N\rangle_{x,x',t}$. We now derive an equation for the latter. From (4.6) and (6.2) we first derive an equation for the fluctuation δN:

$$\left[\frac{\partial}{\partial t} + \boldsymbol{v}\cdot\frac{\partial}{\partial \boldsymbol{r}} + \boldsymbol{F}(\boldsymbol{r},t)\cdot\frac{\partial}{\partial \boldsymbol{p}}\right]\delta N(x,t) + \delta \boldsymbol{F}\cdot\frac{\partial}{\partial \boldsymbol{p}} n f_1(x,t)$$
$$= -\frac{\partial}{\partial \boldsymbol{p}}\cdot\left[\delta \boldsymbol{F}(\boldsymbol{r},t)\,\delta N(x,t) - \langle \delta \boldsymbol{F}\,\delta N\rangle_{\boldsymbol{r},x',t}\right]. \tag{6.5}$$

We write a similar equation for the function $\delta N(x',t)$. We multiply the first equation by $\delta N(x',t)$ and the second by $\delta N(x,t)$, we average both equations and add them term by term. As a result we find the equation for $\langle \delta N\delta N\rangle_{x,x',t}$ which, using (1.8), can be written as follows:

$$\hat{L}_{x,x'}\,\langle \delta N\,\delta N\rangle_{x,x',t} + \langle \delta \boldsymbol{F}\;\delta N\rangle_{\boldsymbol{r},x',t}\cdot\frac{\partial}{\partial \boldsymbol{p}} n f_1(x,t)$$
$$+\langle \delta N\,\delta \boldsymbol{F}\rangle_{x,\boldsymbol{r}',t}\cdot\frac{\partial}{\partial \boldsymbol{p}'}\, n f_1(x',t)$$
$$= -\frac{\partial}{\partial \boldsymbol{p}}\cdot\langle \delta \boldsymbol{F}\;\delta N\;\delta N\rangle_{\boldsymbol{r},x,x',t} - \frac{\partial}{\partial \boldsymbol{p}'}\langle \delta N\,\delta \boldsymbol{F}\;\delta N\rangle_{x,\boldsymbol{r}',x',t}\,. \tag{6.6}$$

This equation is not closed either, as it involves higher moments of the fluctuations δN, $\delta \boldsymbol{F}$. Thus, we obtain a hierarchy of equations for the moments, analogous

to the BBGKY hierarchy for the distribution functions.

We now consider the first moment and the second moment approximations and relate the latter to the polarization approximation.

In the first moment approximation, the right-hand side of Eq. (6.2) is neglected and we find a closed set of equations for the average phase density $\langle N(x,t)\rangle = n f_1(x,t)$ and for the average force $F(r,t)$:

$$\frac{\partial f_1}{\partial t} + v \cdot \frac{\partial f_1}{\partial r} + F(r,t) \cdot \frac{\partial f_1}{\partial p} = 0$$

$$F = F_0 - n \frac{\partial}{\partial r} \cdot \int dx' \, \Phi(|r - r'|) f_1(x',t) \ . \tag{6.7}$$

This set of equations can also be written as a single nonlinear equation for the function f_1. Equation (6.7) is called the self-consistent field equation, because the distribution function f_1 is defined through the force, and the latter depends on the distribution function.

The self-consistent field approximation provides a good description of processes in a plasma, whenever the characteristic size of the system is much larger than the mean free path. Under these conditions, the plasma is called 'collisionless'. Equation (6.7) is also call the Vlasov equation, after the name of its first discoverer †.

In ordinary gases, the second term of the average force (6.7) is negligibly small, hence the first equation (6.7) reduces to

$$\frac{\partial f_1}{\partial t} + v \cdot \frac{\partial f_1}{\partial r} + F_0(r,t) \cdot \frac{\partial f_1}{\partial p} = 0 \ . \tag{6.8}$$

This equation describes the processes in an extremely dilute gas, when the characteristic dimension of the system is much larger than the mean free path.

Equations (6.7), (6.8), corresponding to the first moment approximation, cannot describe dissipative (or relaxation) processes. They cannot be used for the study of the approach to equilibrium or of the irreversible processes such as viscosity, heat conduction or diffusion. This can be seen by defining the entropy as follows

$$S(t) = - k_B n \int dx \ f_1(x,t) \ \ln f_1(x.t) \ . \tag{6.9}$$

When the rate of change of the function f_1 is determined by Eqs (6.7) or (6.8), the entropy remains constant in time:

$$dS/dt = 0 \tag{6.10}$$

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† **Equation (6.7) has the same form as the exact microscopic equation (4.6). This property is well known in quantum theory (see section 72). In classical theory, this fact was pointed out by Vlasov. (See reference [3], section 13.)**

We now consider the second moment approximation. From (6.6), neglecting the terms involving third moments, we find

$$\hat{L}_{x,x'}\langle \delta N\, \delta N\rangle_{x,x',t} + \langle \delta F\ \delta N\rangle_{\boldsymbol{r},x',t} \cdot \frac{\partial}{\partial \boldsymbol{p}} n f_1(x,t) + \langle \delta N\ \delta \boldsymbol{F}\rangle_{x,\boldsymbol{r}',t} \cdot \frac{\partial}{\partial \boldsymbol{p}'} n f_1(x',t) = 0 . \tag{6.11}$$

This equation can also be obtained directly from (6.5) — in which the nonlinear terms in the fluctuations are neglected — by multiplication by δN and use of the equation for $\delta \boldsymbol{F}$:

$$\left[\frac{\partial}{\partial t} + \boldsymbol{v} \cdot \frac{\partial}{\partial \boldsymbol{r}} + \boldsymbol{F}(\boldsymbol{r},t) \cdot \frac{\partial}{\partial \boldsymbol{p}}\right] \delta N(x,t) + \delta \boldsymbol{F}(\boldsymbol{r},t) \cdot \frac{\partial}{\partial \boldsymbol{p}} n f_1(x,t) = 0$$

$$\delta \boldsymbol{F}(\boldsymbol{r},t) = -\frac{\partial}{\partial \boldsymbol{r}} \int dx'\ \Phi(|\boldsymbol{r}-\boldsymbol{r}'|)\ \delta N(x',t) . \tag{6.12}$$

The second moment approximation is not quite consistent; indeed, we see from (5.10) that the third central moments contain terms involving the functions f_1 and g_2 which define the second moments. Let us keep in Eq. (6.6) the contributions to the third moments which involve f_1 and g_2, under the assumption that $g_2 \ll f_1 f_1$. We then find, instead of (6.11) the following more general equation:

$$\hat{L}_{x,x'}\langle \delta N\, \delta N\rangle_{x,x',t} + \langle \delta F\ \delta N\rangle_{\boldsymbol{r},x',x} \cdot \frac{\partial}{\partial \boldsymbol{p}} n f_1(x,t) + \langle \delta N\ \delta \boldsymbol{F}\rangle_{x,\boldsymbol{r}',t} \cdot \frac{\partial}{\partial \boldsymbol{p}'} n f_1(x',t) = \hat{L}_{x,x'}\langle \delta N\, \delta N\rangle^{\text{source}}_{x,x',t} . \tag{6.13}$$

Here we introduced the notation:

$$\langle \delta N\ \delta N\rangle^{\text{source}}_{x,x',t} = n\left[\delta(x-x')\, f_1(x,t) - \frac{1}{V} f_1(x,t)\, f_1(x',t)\right]. \tag{6.14}$$

In the limiting case when $N \to \infty$, $V \to \infty$, $n = \text{const}$,

$$\langle \delta N\, \delta N\rangle^{\text{source}}_{x,x',t} = n\, \delta(x-x')\, f_1(x,t) . \tag{6.15}$$

The superscript 'source' stresses the fact that these expressions do not depend on second moments, but are entirely determined by the one-particle distribution function f_1.

It is easily seen that Eq. (6.13) is equivalent to Eq. (3.5) determining the correlation function g_2 in the polarization approximation. Indeed, it was obtained under the same assumptions that $g_3 = 0$ and that $g_2 \ll f_1 f_1$.

By using Eq. (6.5) for the phase density $\delta N(x,t)$ we can also construct a hierarchy of equations for the two-time (and many-time) moments. For instance, the equation for $\langle \delta N\, \delta N\rangle_{x,t,x',t'}$ is obtained upon multiplication of (6.5) by

$\delta N(x', t')$ and averaging

$$\hat{L}_x \langle \delta N\, \delta N \rangle_{x,t,x',t'} + \langle \delta \boldsymbol{F}\, \delta N \rangle_{\boldsymbol{r},t,x',t'} \cdot \frac{\partial}{\partial \boldsymbol{p}} n f_1(x,t)$$
$$= -\frac{\partial}{\partial \boldsymbol{p}} \cdot \langle \delta \boldsymbol{F}\, \delta N\, \delta N \rangle_{\boldsymbol{r},x,t;x',t'} \,. \tag{6.16}$$

This equation is not closed either, as it involves the third two-time moment. In the polarization approximation, i.e., when $g_3(x, x'', t; x', t') = 0$, $g_2(x,t,x',t') \ll f_1(x,t)\, f_1(x',t')$, Eq. (6.16) reduces to:

$$\hat{L}_x \langle \delta N \delta N \rangle_{x,t,x',t'} + \langle \delta \boldsymbol{F}\, \delta N \rangle_{\boldsymbol{r},t,x',t'} \cdot \frac{\partial}{\partial \boldsymbol{p}} n f_1(x,t)$$
$$= \hat{L}_x \langle \delta N\, \delta N \rangle^{\text{source}}_{x,t,x',t'} \tag{6.17}$$

with the notation

$$\langle \delta N\, \delta N \rangle^{\text{source}}_{x,t,x',t'} = n \left\{ \boldsymbol{F}_1(x,t,x',t') - V^{-1} f_1(x,t)\, f_1(x',t') \right\} . \tag{6.18}$$

In the limit $N \to \infty$, $V \to \infty$, $n = \text{const}$,

$$\langle \delta N\, \delta N \rangle^{\text{source}}_{x,t,x',t'} = n\, F_1(x,t,x',t') \,. \tag{6.19}$$

Equation (6.17) must be solved with the initial condition:

$$\langle \delta N\, \delta N \rangle_{x,t,x',t'} \Big|_{t=t'} = \langle \delta N\, \delta N \rangle_{x,x',t} \,. \tag{6.20}$$

Thus, in order to solve (6.17), we must know the solution of (6.13) for the one-time moment. We will see that, under certain conditions, such a solution can indeed be found.

Let us note that the source $\langle \delta N\, \delta N \rangle^{\text{source}}_{x,t,x',t'}$ in Eq. (6.17) is defined in terms of the two-time one-particle distribution $F_1(x,t,x',t')$. In order to obtain an equation for this function, we introduce the two-time phase density:

$$N(x,t,x',t') = \sum_{1 \leqslant i \leqslant N} \delta[x - x_i(t)]\, \delta[x' - x'(t')] \,. \tag{6.21}$$

It follows from (5.11) that

$$\langle N(x,t,x',t') \rangle = n\, F_1(x,t,x',t') \,. \tag{6.22}$$

The equation for the two-time phase density is

$$\left\{ \frac{\partial}{\partial t} + \boldsymbol{v} \cdot \frac{\partial}{\partial \boldsymbol{r}} + \boldsymbol{F}^{\mathrm{M}}(\boldsymbol{r},t) \cdot \frac{\partial}{\partial \boldsymbol{p}} \right\} N(x,t,x',t') = 0 \,. \tag{6.23}$$

The microscopic force $\boldsymbol{F}^{\mathrm{M}}(\boldsymbol{r},t)$ is defined by (4.5). Averaging this equation and using (6.22), we find :

$$\left\{\frac{\partial}{\partial t} + \boldsymbol{v}\cdot\frac{\partial}{\partial \boldsymbol{r}} + \boldsymbol{F}(\boldsymbol{r},t)\cdot\frac{\partial}{\partial \boldsymbol{p}}\right\} F_1(x,t,x',t')$$

$$= - n^{-1}\frac{\partial}{\partial \boldsymbol{p}}\cdot\left\langle \delta \boldsymbol{F} \quad \delta N\right\rangle_{\boldsymbol{r},t,x',t'} \equiv I(x,t,x',t'). \tag{6.24}$$

This equation must be solved with the initial condition:

$$F_1(x,t,x',t')_{t=t'} = \delta(x-x')\, f_1(x,t). \tag{6.25}$$

In deriving Eq. (6.24), we used the identity

$$\left\langle \boldsymbol{F}^{\mathrm{M}} N\right\rangle_{\boldsymbol{r},t;\; x,\, t,x',t'} = \boldsymbol{F}(\boldsymbol{r},t)\left\langle N(x,t,x',t')\right\rangle + \left\langle \delta \boldsymbol{F} \quad \delta N\right\rangle_{\boldsymbol{r},t\,;\,x,t,x',t'} \tag{6.26}$$

where

$$\delta N(x,t,\,x',\,t') = N(x,t,x',t') - \left\langle N(x,t,x',t')\right\rangle$$

The collision integral $I(x,t,x',t')$ in Eq. (6.24) is expressed in terms of the second moment of the fluctuations $\delta N(x'',t)$, $\delta N(x,t,x',t')$. For $N\to\infty$, $V\to\infty$, the latter can be represented as follows:

$$\left\langle \delta N \ \delta N\right\rangle_{x,t,x',t';x'',t}$$

$$= n\,\delta(x-x'')\, F_1(x,t,x',t') + n^2 g_2(x',t,x',t';x'',t)\,. \tag{6.27}$$

This equality follows from the definitions (4.1), (6.21) of the phase densities $N(x,t)$, $N(x,t,x',t')$.

In (6.27), the function $g_2(x,t,x',t';x'',t)$ defines at time t the correlation of two particles at points x, x'', knowing that one of them was at time t' in point x'. We deal here with a triple correlation of two particles. For $t = t'$:

$$g_2(x,t,x',t'\,;\,x'',t) = \delta(x-x')\, g_2(x,x'',t)\ . \tag{6.28}$$

The collision integral in (6.24) only plays a role when $t-t' \sim \tau_{\mathbf{col}}$. For times such that

$$t-t' \ll \tau_{\mathbf{col}}, \tag{6.29}$$

the function $g_2(x,t,x',t'\,;\,x'',t)$ can be neglected. As the first term in (6.27) does not contribute to the collision integral, we may set $I(x,t,x',t') = 0$ in this approximation. We then find, using (6.19):

$$\hat{L}_x\, n\, F_1(x,t,x',t') \equiv \hat{L}_x \left\langle \delta N \quad \delta N\right\rangle^{\mathbf{source}}_{x,t,x',t'} = 0\ . \tag{6.30}$$

Thus, in the polarization approximation, and under the condition (6.29), Eq. (6.17) reduces to

$$\hat{L}_x \left\langle \delta N\, \delta N\right\rangle_{x,t,x',t'} + \left\langle \delta \boldsymbol{F} \quad \delta N\right\rangle_{\boldsymbol{r},t,x',t'}\cdot\frac{\partial}{\partial \boldsymbol{p}}\, n\, f_1(x,t) = 0\ . \tag{6.31}$$

It must be solved with the initial condition (6.20).†

† Equation (6.31) describes short range fluctuations, because of condition (6.29). Its form corresponds to Onsager's hypothesis about the time evolution of fluctuations.

Thus, in order to determine the two-time moments, we must first solve Eq. (6.13). This problem is, in general, complicated. It can however be significantly simplified if we take as a starting point Eq. (6.12) for δN with a source term equivalent to the one of Eq. (6.13), i.e., the following equation:

$$\hat{L}_x\left\{\delta N(x,t) - \delta N^{\text{source}}(x,t)\right\} + \delta \boldsymbol{F}(\boldsymbol{r},t)\cdot\frac{\partial}{\partial \boldsymbol{p}} n f_1(x,t) = 0 . \qquad (6.32)$$

The correlation of the source fluctuations is defined by Eq. (6.30), which we rewrite explicitly

$$\left\{\frac{\partial}{\partial t} + \boldsymbol{v}\cdot\frac{\partial}{\partial \boldsymbol{r}} + \boldsymbol{F}(\boldsymbol{r},t)\cdot\frac{\partial}{\partial \boldsymbol{p}}\right\}\langle \delta N \; \delta N\rangle^{\text{source}}_{x,\,t,\,x',\,t'} = 0 . \qquad (6.33)$$

This equation must be solved with the initial condition:

$$\langle \delta N\,\delta N\rangle^{\text{source}}_{x,\,t,\,x',\,t'}\Big|_{t=t'} =$$

$$= n\left\{\delta(x-x')\, f_1(x,t) - V^{-1} f_1(x,t)\, f_1(x',t)\right\} \qquad (6.34)$$

It will be shown in sections 34 - 39 that by this method the expressions for the two-time and one-time correlations can be obtained quite simply, without solving the integral equation (6.13).

7. TRANSPORT EQUATIONS FOR THE MASS DENSITY, MOMENTUM DENSITY AND KINETIC ENERGY DENSITY

We now consider the balance equations for the mass density $\rho(\boldsymbol{r},t) = mn(\boldsymbol{r},t)$, the momentum density $\rho(\boldsymbol{r},t)\,\boldsymbol{u}(\boldsymbol{r},t)$ and the kinetic energy density $W(\boldsymbol{r},t)$. From Eqs. (4.4) and (5.1) we find the expressions of these quantities in terms of the one-particle distribution function $f_1(x,t)$:

$$\rho(\boldsymbol{r},t) = mn\int d\boldsymbol{p}\; f_1(x,t)$$

$$\rho\boldsymbol{u}(\boldsymbol{r},t) = n\int d\boldsymbol{p}\;\boldsymbol{p}\; f_1(x,t)$$

$$W(\boldsymbol{r},t) = n\int d\boldsymbol{p}\;(\boldsymbol{p}^2/2m)\; f_1(x,t) \qquad (7.1)$$

In order to find the rate of change of these functions, we multiply Eq. (6.2) successively by mn, $n\boldsymbol{p}$, $n\boldsymbol{p}^2/2m$ and integrate over $\boldsymbol{p}$. We consider each of the resulting equations in turn.

Noting that

$$\int d\boldsymbol{p}\,\boldsymbol{F}\cdot(\partial f_1/\partial \boldsymbol{p}) = 0$$

$$mn\int d\boldsymbol{p}\,\boldsymbol{v}\cdot\frac{\partial}{\partial \boldsymbol{r}} f_1 = \frac{\partial}{\partial \boldsymbol{r}}\cdot\rho\boldsymbol{u}$$

and that the integral over the momenta of the right-hand side of Eq. (6.2) vanishes. we find the continuity equation :

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial \boldsymbol{r}} \cdot \rho \boldsymbol{u} = 0 \tag{7.2}$$

which represents the balance of the mass density.

The second equation has the following form:

$$\frac{\partial}{\partial t} \rho u_i + \frac{\partial}{\partial r_j} nm \int d\boldsymbol{p}\ v_i v_j f_1 = m^{-1} \rho F_i + n \int d\boldsymbol{p}\ p_i\ I(x,t) \ . \tag{7.3}$$

The second term of the left-hand side is transformed as follows. The velocity $\boldsymbol{v}$ can be written as $\boldsymbol{v} = \boldsymbol{u} + \delta \boldsymbol{v}$ (where $\delta \boldsymbol{v}$ is the deviation of the velocity $\boldsymbol{v}$ from its average $\boldsymbol{u}$). Thus

$$nm \int d\boldsymbol{p}\ v_i v_j f_1 = \rho\, u_i u_j + P_{ij} \ . \tag{7.4}$$

We introduced here the pressure tensor defined as:

$$P_{ij} = nm \int d\boldsymbol{p}\ \ \delta v_i\ \delta v_j f_1 \ . \tag{7.5}$$

This tensor can, in turn, be represented as follows

$$P_{ij} = \delta_{ij}\, p_{\mathbf{id}} + \pi_{ij}\ , \quad p_{\mathbf{id}} = \tfrac{1}{3} P_{ii} \tag{7.6}$$

where

$$p_{\mathbf{id}} = nm \int d\boldsymbol{p}\ (\delta v)^2 f_1 = (\rho/m)\, k_{\mathbf{B}} T \tag{7.7}$$

is the pressure of the ideal gas, and

$$(\rho/m)\, k_{\mathbf{B}} T = \tfrac{2}{3} n \int d\boldsymbol{p}\, \tfrac{1}{2} m (\delta v)^2 f_1 \tag{7.8}$$

defines the temperature T, $k_{\mathbf{B}}$ being the Boltzmann constant.

The tensor π_{ij} is called the viscous stress tensor, for reasons which will appear more clearly later.

Using these results, we rewrite Eq. (7.3) in the form:

$$\frac{\partial}{\partial t} \rho\, u_i + \frac{\partial}{\partial r_j} \left(\rho\, u_i u_j + \delta_{ij}\, p_{\mathbf{id}} + \pi_{ij} \right) = \frac{\rho}{m} F_i + n \int d\boldsymbol{p}\ p_i\ I \ . \tag{7.9}$$

This is the balance equation for the momentum density.

The last equation in our set is:

$$\frac{\partial}{\partial t} n \int d\boldsymbol{p}\ (p^2/2m) f_1 + \frac{\partial}{\partial \boldsymbol{r}} \cdot n \int d\boldsymbol{p}\ \boldsymbol{v}\ \frac{p^2}{2m} f_1 = \frac{\rho}{m} \boldsymbol{F} \cdot \boldsymbol{u} + n \int d\boldsymbol{p}\ \frac{p^2}{2m}\ I \ . \tag{7.10}$$

Using the definition (7.8) for the temperature, and the definition of the average velocity, the first term in the left-hand side can be written as:

$$\frac{\partial}{\partial t} \left(\tfrac{1}{2} \rho \boldsymbol{u}^2 + \tfrac{3}{2} \frac{\rho}{m} k_{\mathbf{B}} T \right) \ . \tag{7.11}$$

We represent $\boldsymbol{v}$ in the second term of Eq. (7.10) as $\boldsymbol{v} = \boldsymbol{u} + \delta \boldsymbol{v}$, we use the definitions of $\boldsymbol{u}$, T, P_{ij}, π_{ij}, $p_{\mathbf{id}}$ and introduce the notation $\boldsymbol{S}$ for the heat flow vector:

$$\boldsymbol{S} = n\int d\boldsymbol{p}\ \delta\boldsymbol{v}\ \tfrac{1}{2}\ m(\delta v)^2\, f_1\ . \tag{7.12}$$

As a result, the second term can be written as:

$$\frac{\partial}{\partial r_i}\left\{u_i\left(\tfrac{1}{2}\,\rho\, u^2 + \tfrac{3}{2}\,\frac{\rho}{m}\,k_{\mathbf{B}}T + p_{\mathbf{id}}\right) + \pi_{ij}\,u_j + S_i\right\}\ . \tag{7.13}$$

Substituting (7.13) and (7.11) into (7.10) we find the energy balance equation:

$$\frac{\partial}{\partial t}\left\{\tfrac{1}{2}\,\rho u^2 + \tfrac{3}{2}\,k_{\mathbf{B}}T\right\} + \frac{\partial}{\partial r_i}\left\{u_i\left(\tfrac{1}{2}\,\rho\,u^2 + \tfrac{3}{2}\,\frac{\rho}{m}\,k_{\mathbf{B}}T + p_{\mathbf{id}}\right)\right.$$

$$\left. + \pi_{ij}\ u_j + S_i\right\} = \frac{\rho}{m}\,\boldsymbol{F}\cdot\boldsymbol{u} + n\int d\boldsymbol{p}\ \frac{p^2}{2m}\ I\ . \tag{7.14}$$

We call attention upon the fact that the first term in this equation contains the time derivative of the sum of the macroscopic kinetic energy density $\rho u^2/2$ and of the internal energy density of the ideal gas. Actually, for a non-ideal gas we would expect to find also the internal energy of the real gas under the time derivative. We will see that this contribution, related to the particle interactions, is contained in the last term of the right-hand side of (7.14). This term will also yield a modification of the energy flow due to the interactions. An analogous role is played by the last term of Eq. (7.9): it defines the contribution of the interactions to the momentum flow. All these facts will be proved in chapter 2.

The set of five balance equations (7.2), (7.9) and (7.14) is not closed, because besides the five functions ρ, $\boldsymbol{u}$, T, it also involves the functions

$$\pi_{ij}\,,\quad \boldsymbol{S}\,,\quad \int d\boldsymbol{p}\ \boldsymbol{p} I\,,\quad \int d\boldsymbol{p}\ \frac{p^2}{2m}\ I\ . \tag{7.15}$$

The lack of closure of the balance equations is due, partly, to the hierarchical structure of the original equation (6.2), which is only the first of the chain of equations for the distribution functions. In the next chapter, a closed equation for the one-particle distribution (the Boltzmann kinetic equation for the non-ideal gas) will be derived in the binary collision approximation. In this equation, the interactions play a double role. They define the terms describing the dissipative processes, but they also contribute to the non-dissipative properties of the gas, such as the internal energy, the pressure, etc.

If the contribution of the interactions to the non-dissipative properties is neglected (i.e. the approximation of the ideal gas), the collision integral has the properties:

$$\int d\boldsymbol{p}\,\boldsymbol{p}\ I = 0\,,\qquad \int d\boldsymbol{p}\ \frac{p^2}{2m}\ I = 0\ . \tag{7.16}$$

as a result, there remain only two additional quantities in the balance equations:

π_{ij} and $\boldsymbol{S}$ (there are altogether eight scalar functions, as the tensor π_{ij} is symmetric and traceless). In the approximation in which the characteristic dimension of the system L is much larger than the mean free path, the Boltzmann equation yields the following expressions of these quantities in terms of ρ, $\boldsymbol{u}$, T:

$$\pi_{ij} = -\eta \left(\frac{\partial}{\partial r_j} u_i + \frac{\partial}{\partial r_i} u_j - \tfrac{2}{3} \delta_{ij} \frac{\partial}{\partial r_k} u_k \right) \tag{7.17}$$

$$\boldsymbol{S} = - \kappa \frac{\partial}{\partial \boldsymbol{r}} T . \tag{7.18}$$

Here η is the viscosity coefficient and κ the thermal conductivity. The following expressions are obtained for these coefficients [2, 3, 5, 9, 12] :

$$\eta = \frac{5m}{16 \pi^{\frac{1}{2}} r_0^2} \left(\frac{k_{\mathrm{B}} T}{m} \right)^{\frac{1}{2}}$$

$$\kappa = \frac{75}{32} \frac{k_{\mathrm{B}}}{\pi^{\frac{1}{2}} r_0^2} \left(\frac{k_{\mathrm{B}} T}{m} \right)^{\frac{1}{2}} . \tag{7.19}$$

If Eqs (7.16) - (7.18) are substituted into the balance equations (7.2), (7.9) and (7.14) these become a closed set of equations for the functions ρ, $\boldsymbol{u}$, T: the hydrodynamical equations of the ideal gas. We will see in chapter 2 that the Boltzmann equation for the nonideal gas provides more general hydrodynamical equations, taking into account the contribution of the interactions to the thermodynamic functions.

CHAPTER 2

The Boltzmann Kinetic Equation for Nonideal Gases

8. SOME RESULTS OF THE EQUILIBRIUM STATISTICAL THEORY OF NONIDEAL GASES

We recall here a few results of the statistical theory of gases, which will be used below. Consider the canonical Gibbs distribution

$$f_N(x,a,T) = \exp \frac{F - H(x,a)}{k_B T} \tag{8.1}$$

where a is an external parameter, and F is the free energy, related to the partition function Z:

$$F = -k_B T \ln Z$$

$$Z = \int dx \; exp\left(-\frac{H(x,a)}{k_B T}\right) .$$

From (8.1) we can obtain the Bogolyubov hierarchy of equations for the equilibrium distribution function. When the Hamiltonian is of the form (1.1), the momentum distribution function of each particle is Maxwellian; we may therefore restrict attention to the distribution of positions: $f_1(\mathbf{r}_1)$, $f_2(\mathbf{r}_1,\mathbf{r}_2),\ldots$ The first two equations of this hierarchy are [4,5] (they can be obtained from Eqs (1.19) and (1.20)):

$$\frac{\partial}{\partial \mathbf{r}_1} f_1 + \frac{1}{k_B T}\frac{\partial u}{\partial \mathbf{r}_1} f_1 = -\frac{n}{k_B T}\int d\mathbf{r}_2 \left(\frac{\partial}{\partial \mathbf{r}_1}\Phi_{12}\right) f_2 \tag{8.2}$$

$$\frac{\partial}{\partial \mathbf{r}_1} f_2 + \frac{1}{k_B T}\frac{\partial u}{\partial \mathbf{r}_1} f_2 + \frac{1}{k_B T}\frac{\partial \Phi_{12}}{\partial \mathbf{r}_1} f_2 = -\frac{n}{k_B T}\int d\mathbf{r}_3 \frac{\partial \Phi_{13}}{\partial \mathbf{r}_1} f_3 \tag{8.3}$$

where u is the potential of the external force. In the case $u=0$, $f_2 = f_2(|\mathbf{r}_1 - \mathbf{r}_2|)$ and we obtain from Eq. (8.2):

KTNG - B*

$$f_1(\mathbf{r}_1) = 1 , \quad V^{-1}\int d\mathbf{r}_1 \; f_1 = 1 . \tag{8.4}$$

These equations describe a spatially homogeneous distribution.

From Eq. (8.4) follow the relations between the distribution functions and the correlation functions in the form:

$$f_2(1,2) = 1 + g_2(1,2)$$

$$f_3(1,2,3) = 1 + g_2(1,2) + g_2(2,3) + g_2(1,3) + g_3(1,2,3) \tag{8.5}$$

where $g_2(i,j) = g_2(|\boldsymbol{r}_i - \boldsymbol{r}_j|)$.

In the binary collision approximation (i.e. to first order in the density parameter) we set $g_3 = 0$ in (8.5). In this case, the right-hand side of Eq. (8.3) vanishes:

$$\int d\boldsymbol{r}_3 \frac{\partial \Phi_{13}}{\partial \boldsymbol{r}_1} \left\{1 + g_2(1,3) + g_2(1,2) + g_2(2,3)\right\} = 0 \tag{8.6}$$

The first two terms vanish because the integral is odd in the variable $\boldsymbol{r}_{13}$. The remaining two terms only contribute when the three particles interact simultaneously: they must be neglected in the present approximation. As a result the equation for f_2 reduces to:

$$\frac{\partial f_2}{\partial \boldsymbol{r}_1} + \frac{1}{k_B T} \frac{\partial \Phi_{12}}{\partial \boldsymbol{r}_1} f_2 = 0 . \tag{8.7}$$

Hence

$$f_2 = C \exp(-\Phi_{12}/k_B T) . \tag{8.8}$$

The constant C is obtained from the normalization condition (1.14)

$$C = \left\{V^{-2} \int d\boldsymbol{r}_1 \, d\boldsymbol{r}_2 \, \exp(-\Phi_{12}/k_B T)\right\}^{-1}$$

$$= \left\{1 + 4\pi V^{-1} \int_0^\infty dr\, r^2 \left[\exp(-\Phi/k_B T) - 1\right]\right\}^{-1} = 1 . \tag{8.9}$$

Indeed, the contribution of the second term is of order r_0^3/V and vanishes in the limit $V \to \infty$. We thus obtain the following expressions for f_2, g_2 in the binary collision approximation:

$$f_2 = \exp(-\Phi_{12}/k_B T)$$

$$g_2 = \exp(-\Phi_{12}/k_B T) - 1 . \tag{8.10}$$

We now calculate the main thermodynamic functions in this approximation. We therefore look for expressions of the internal energy and for the pressure in terms of f_1 and f_2 .

Averaging the Hamiltonian (1.1) with the weight f_N and using the definitions (1.12), (1.13) we find the following expression for the internal energy density:

$$U = \frac{\langle H \rangle}{V} = n \int d\boldsymbol{p} \frac{p^2}{2m} f_1 + \frac{n^2}{2} V^{-1} \int dx \, dx' \, \Phi(|\boldsymbol{r} - \boldsymbol{r}'|) f_2(x, x') \tag{8.11}$$

Hence, using for f_1 the Maxwell distribution, and for f_2 Eq. (8.10):

$$U = \frac{3}{2} n k_B T + 2\pi n^2 \int_0^\infty dr\, r^2 \Phi(r) \exp\left[-\Phi(r)/k_B T\right] \equiv U_{id} + \Delta U \tag{8.12}$$

We denote by ΔU the contribution of the interactions to the internal energy density.

The relation between the pressure and the two-particle distribution follows from the well-known expression:

$$p = -\left(\frac{\partial F}{\partial V}\right)_T = k_B T \left(\frac{\partial \ln Z}{\partial V}\right)_T \; ; \tag{8.13}$$

It has the following form: [4,5,9]

$$p = n k_B T - \frac{2}{3} \pi n^2 \int_0^\infty dr\, r^3 \frac{d\Phi(r)}{dr} f_2(r) \equiv p_{id} + \Delta p \; . \tag{8.14}$$

Substituting Eq. (8.10) and integrating by parts we obtain:

$$\Delta p = -k_B T \cdot 2\pi n^2 \int_0^\infty dr\, r^2 \left[\exp(-\Phi/k_B T) - 1\right] . \tag{8.15}$$

From Eqs (8.13) and (8.15) follows the expression of ΔF, the contribution of the interactions to the free energy:

$$(\Delta F/V) = -k_B T \cdot 2\pi n^2 \int_0^\infty dr\, r^2 \left[\exp(-\Phi/k_B T) - 1\right] \; . \tag{8.16}$$

From (8.12) and (8.16) we finally obtain the contribution of the interactions to the entropy :

$$\begin{aligned} \Delta S = {} & 2\pi n^2 k_B \int dr\, r^2 (\Phi/k_B T) \exp(-\Phi/k_B T) \\ & + 2\pi n^2 k_B \int dr\, r^2 \left[\exp(-\Phi/k_B T) - 1\right] \; . \end{aligned} \tag{8.17}$$

The results obtained above are valid for every model of a monatomic gas. Let us consider as an example the model of weakly attracting hard spheres, for which the interaction potential is :

$$\Phi(r) = \begin{cases} 0 & \text{for} \quad r \leq r_0 \\ \Phi(r) < 0 & \text{for} \quad r > r_0 \end{cases} \tag{8.18}$$

with

$$|\Phi(r)|/k_B T \ll 1 \; .$$

We substitute this potential into (8.12), (8,15) and (8.17) and introduce the following notation for the van der Waal's constants b and a :

$$b = \frac{2}{3} \pi r_0^3$$

$$a = 2\pi \int_{r_0}^\infty dr\, r^2 |\Phi(r)| \tag{8.19}$$

The quantities ΔU, Δp, ΔS can then be expressed as follows, to first order in $|\Phi|/k_BT$:

$$\Delta U = -n^2 a, \qquad \Delta p = n^2 k_B T\left(b - \frac{a}{k_B T}\right), \qquad \Delta S = -k_B n^2 b \tag{8.20}$$

From these formulae follows that the contribution of the interactions to the thermodynamic functions are small, of order $nr_0^3 = \varepsilon$ compared to the ideal part.

We now consider the fluctuations of the particle density in a volume element ΔV which is much smaller than V but still contains many particles. We return to formula (5.6) which connects $\langle \delta N\ \delta N\rangle_{x,x',t}$ and $g_2(x,x',t)$, and integrate both sides over the momenta $\boldsymbol{p}, \boldsymbol{p}'$. Using the properties:

$$\int d\boldsymbol{p}\ \delta N(x,t) = \delta n(\boldsymbol{r},t)$$

$$\int d\boldsymbol{p}\, d\boldsymbol{p}'\, g_2(x,x') = g_2(\boldsymbol{r},\boldsymbol{r}')$$

$$(N/V)\int d\boldsymbol{p}\ f_1 = n(\boldsymbol{r})$$

we derive the relation between the one-time correlation of the fluctuations $\delta n(\boldsymbol{r})$ and the spatial correlation function $g_2(\boldsymbol{r},\boldsymbol{r}')$;

$$\langle \delta \boldsymbol{n}\ \delta \boldsymbol{n}\rangle_{\boldsymbol{r},\boldsymbol{r}'} = \frac{N(N-1)}{V^2}\, g_2(\boldsymbol{r},\boldsymbol{r}') + [\delta(\boldsymbol{r}-\boldsymbol{r}')\, n(\boldsymbol{r}) - N^{-1}\, n(\boldsymbol{r})\, n(\boldsymbol{r}')]. \tag{8.21}$$

We shall denote by

$$N_{\Delta V} = \int_{\Delta V} d\boldsymbol{r}\ n(\boldsymbol{r}) \tag{8.22}$$

the average number of particles in the volume ΔV, and by $\delta N_{\Delta V}$ the corresponding fluctuation. We now integrate both sides of (8.21) over $\boldsymbol{r},\boldsymbol{r}'$ in the domain ΔV and make use of the expressions:

$$(N/V)^2\int_{\Delta V} d\boldsymbol{r}\ d\boldsymbol{r}'\ g_2(\boldsymbol{r},\boldsymbol{r}') = n\, N_{\Delta V}\int d\boldsymbol{r}\ g_2(\boldsymbol{r}),$$

$$\int d\boldsymbol{r}\ d\boldsymbol{r}'\ [\delta(\boldsymbol{r}-\boldsymbol{r}')\, n(\boldsymbol{r}) - N^{-1}\ n(\boldsymbol{r})\ n(\boldsymbol{r}')] = N_{\Delta V}[1-(N_{\Delta V}/N)] = N_{\Delta V}. \tag{8.23}$$

From (8.21) - (8.23) we derive the following formula for the fluctuation of the number of particles in the volume ΔV:

$$\frac{\langle \delta^2 N_{\Delta V}\rangle}{N_{\Delta V}^2} = \frac{1}{N_{\Delta V}}\left\{1 + n\int d\boldsymbol{r}\ \ g_2(\boldsymbol{r})\right\} \tag{8.24}$$

The second term on the right-hand side represents the role of the interactions. It involves the binary correlation function.

We consider here again the approximation of binary collisions, in which case Eq. (8.10) yields

$$\frac{\langle \delta^2 N_{\Delta V} \rangle}{N^2_{\Delta V}} = \frac{1}{N_{\Delta V}} \left\{ 1 + 4\pi n \int dr\, r^2 \left[\exp\left(-\Phi/k_B T\right) - 1 \right] \right\}. \tag{8.25}$$

For the model of weakly attracting spheres, with the potential (8.18), we find:

$$\frac{\langle \delta^2 N_{\Delta V} \rangle}{N^2_{\Delta V}} = \frac{1}{N_{\Delta V}} \left\{ 1 + 2n \left(\frac{a}{k_B T} - b \right) \right\} . \tag{8.26}$$

As in the case of the pressure, the interaction term results from a competition between the effects of the attraction and of the repulsion, characterized, respectively, by the van der Waal's constants a and b .

The expressions obtained above represent the zeroth and first order terms in an expansion of the thermodynamic functions as a series in powers of the density, the so-called virial expansion. Analogous virial expansions exist for the functions f_2 , g_2. Equations (8.10) define the first terms of these expansions. From the hierarchy of equations for the distribution functions, the functions f_2, g_2 can be determined to arbitrary order in the density parameter [4,5]. Thus, the first two terms in the expansion of f_2 , g_2 are

$$f_2 = \exp\left(-\Phi_{12}/k_B T\right) \left\{ 1 + n \int d\mathbf{r}_3 \left[\exp(-\Phi_{13}/k_B T) - 1 \right] \left[\exp\left(-\Phi_{23}/k_B T\right) - 1 \right] \right.$$

$$g_2 = f_2 - 1 . \tag{8.27}$$

Using these expressions, the first three terms of the virial expansion of the thermodynamic functions can be derived.

The use of higher-order terms in the virial expansion is less effective. These terms are very complicated, and do not provide a convenient method for calculating the thermodynamic properties of dense gases. These properties are computed either by numerical methods (the method of molecular dynamics, the Monte Carlo method), or by solving model integral equations for the pair correlation function [33 , 34].

9. NONLINEAR INTEGRAL EQUATIONS FOR THE TWO-PARTICLE CORRELATION FUNCTION

A certain number of non-linear integral equations have been derived for the correlation function $g_2(r)$.† Historically, the first of these was the so-called Kirkwood-Bogolyubov-Born-Green equation. It can be obtained from the second equation of the Bogolyubov hierarchy (8.3) by substituting for f_3 the superposition approximation due to Kirkwood. For spatially homogeneous gases, the superposition

† **A thorough treatment of the integral equations is to be found in ref. [70]. See also [67 , 69]. (Translator.)**

approximation is:

$$f_3(\boldsymbol{r}_1, \boldsymbol{r}_2, \boldsymbol{r}_3) = f_2(\boldsymbol{r}_1 - \boldsymbol{r}_2)\, f_2(\boldsymbol{r}_2 - \boldsymbol{r}_3)\, f_2(\boldsymbol{r}_3 - \boldsymbol{r}_1) \tag{9.1}$$

In a series of papers [36 , 61] it was shown that the Kirkwood-Bogolyubov-Born-Green equation leads in many cases to a qualitative description of the properties of dense gases and of simple liquids, but is insufficient for numerical calculations.

Percus and Yevick derived another integral equation for the description of the properties of liquids, which gives better results. This equation was first derived on the basis of a physical picture of the character of the collective motions in a liquid. In more recent work [33 , 34] a more fundamental justification of the Percus-Yevick equation was obtained.

Formally, the Percus-Yevick equation can be obtained from the well-known Ornstein-Zernike equation [33 , 34] :

$$g_2(\boldsymbol{r}_1, \boldsymbol{r}_2) = c(\boldsymbol{r}_1, \boldsymbol{r}_2) + n \int d\boldsymbol{r}_3\; c(\boldsymbol{r}_1, \boldsymbol{r}_3)\, g_2(\boldsymbol{r}_3, \boldsymbol{r}_2) \tag{9.2}$$

which in fact, provides a definition of the so-called direct correlation function $c(\boldsymbol{r}_1, \boldsymbol{r}_2)$ in terms of the pair correlation function g_2 . In order to derive the Percus-Yevick equation, we need a second relation between the functions c and g_2:

$$c(\boldsymbol{r}_1, \boldsymbol{r}_2) = [1 + g_2(\boldsymbol{r}_1, \boldsymbol{r}_2)][1 - \exp(-\Phi_{12}/k_B T)] . \tag{9.3}$$

From (9.2) and (9.3) we obtain a closed equation for g_2 or, equivalently, for $f_2 = 1 + g_2$:

$$f_2(1,2) = \exp(-\Phi_{12}/k_B T) \tag{9.4}$$
$$\left\{1 + n\int d\boldsymbol{r}_3\; \exp(\Phi_{13}/k_B T)\, f_2(1,3)[\exp(-\Phi_{13}/k_B T) - 1][f_2(2,3) - 1]\right\}$$

Another model equation is provided by the 'hypernetted-chain' approximation. This model can also be obtained from the Ornstein-Zernike equation, by using instead of (9.3), a different relation between c and g_2 :

$$c(1,2) = f_2(1,2) - 1 - \ln f_2(1,2) - \Phi_{12}/k_B T \tag{9.5}$$

Both expression (9.3) and (9.4) satisfy the general conditions for the asymptotic behaviour of the functions f_2 and c for $r \to \infty$.

In ref. [35] a still different type of equation was used:

$$f_2(1,2) = \exp(-\Phi_{12}/k_B T)\left\{1 + n\int d\boldsymbol{r}_3 [f_2(1,3) - 1][f_2(2,3) - 1]\right\} . \tag{9.6}$$

It is more symmetrical than the Percus-Yevick equation, but has a different asymptotic behaviour at large distances.

All the proposed integral equations yield exactly the first two terms of

the virial expansion of f_2 but give different contributions to higher order in the density parameter.

10. THE BOLTZMANN KINETIC EQUATION FOR THE SPATIALLY HOMOGENEOUS IDEAL GAS

In section 2 a closed set of two equations was obtained for the functions f_1, f_2 in the binary collision approximation. For a spatially homogeneous gas it simplifies to the following form (see (2.13),(2.20)):

$$\frac{\partial}{\partial t} f_1(\boldsymbol{p}_1, t) = n\int dx_2 \frac{\partial \Phi_{12}}{\partial \boldsymbol{r}_1} \cdot \frac{\partial}{\partial \boldsymbol{p}_1} g_2(\boldsymbol{r}_1 - \boldsymbol{r}_2, \boldsymbol{p}_1, \boldsymbol{p}_2, t) \equiv I(\boldsymbol{p}_1, t) \tag{10.1}$$

$$\left\{\frac{\partial}{\partial t} + \boldsymbol{v}_1 \cdot \frac{\partial}{\partial \boldsymbol{r}_1} + \boldsymbol{v}_2 \cdot \frac{\partial}{\partial \boldsymbol{r}_2} - \frac{\partial \Phi_{12}}{\partial \boldsymbol{r}_1} \cdot \frac{\partial}{\partial \boldsymbol{p}_1} - \frac{\partial \Phi_{12}}{\partial \boldsymbol{r}_2} \cdot \frac{\partial}{\partial \boldsymbol{p}_2}\right\} f_2(\boldsymbol{r}_1 - \boldsymbol{r}_2, \boldsymbol{p}_1, \boldsymbol{p}_2, t)$$
$$= \frac{\partial}{\partial t} f_1(\boldsymbol{p}_1, t) f_1(\boldsymbol{p}_2, t) \tag{10.2}$$

We now consider additional assumptions, by means of which a single closed equation can be derived for the one-particle distribution f_1: the Boltzmann kinetic equation.†

The solution of Eq. (10.2) can be represented in the form of a sum of two terms, which are, respectively, the solutions of the homogeneous and of the inhomogeneous equations. The former depends on the initial value of the distribution function f_2, and (as $f_2 = f_1 f_1 + g_2$) also on the initial value of the correlations g_2.

The first additional assumption for deriving the Boltzmann equation consists of the introduction of Bogolyubov's condition of complete weakening of the initial correlations.

In fact, for deriving kinetic equations, we may only neglect those initial correlations for which the correlation time $\tau_{\mathbf{cor}}$ is much smaller than the relaxation time of f_1, which in turn is of the order of the mean free path $\tau_{\mathbf{rel}} = l/v_{\mathbf{T}}$. Thus, we may neglect the correlations for which

$$\tau_{\mathbf{cor}} \ll \tau_{\mathbf{rel}} \tag{10.3}$$

Thus, the condition of complete weakening of the initial correlations corresponds to the assumption that the long-living correlations (with $\tau_{\mathbf{cor}} \gg \tau_{\mathbf{rel}}$) do not play an essential role in kinetic theory. We shall see later that this is not always true.

By using the condition of the weakening of initial correlations, the solution of the equation for f_2 is defined only in terms of the product

$$f_1(\boldsymbol{p}_1, t-\tau) f_1(\boldsymbol{p}_2, t-\tau)$$

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† See also refs [67, 68]. (Translator.)

where τ is some positive time. Thus, the function f_2 depends on the value of f_1 at an earlier time: there appears a retardation effect.

The second assumption in the derivation of the Boltzmann equation is the complete neglect of the time retardation. In other words, one assumes that the distribution function f_2 at time t is defined by the value of the function f_1 at the same time t.

In this approximation, the right-hand side of (10.2) can be neglected because it is proportional to the time derivative of f_1. We may thus use simply the homogeneous equation (10.2). By using these approximations, the solution of Eq. (10.2) reduces to

$$f_2(\boldsymbol{r}_1, \boldsymbol{r}_2; \boldsymbol{p}_1, \boldsymbol{p}_2, t) = f_1(\boldsymbol{P}_1(-\infty), t)\, f_1(\boldsymbol{P}_2(-\infty), t)$$
$$\equiv S_{-\infty}^{(2)} f_1(\boldsymbol{p}_1, t)\, f_1(\boldsymbol{p}_2, t) . \tag{10.4}$$

Here

$$\boldsymbol{P}_j(-\infty) \equiv \lim_{\tau\to\infty} \boldsymbol{P}_j(-\tau)$$

is the initial value of the momentum of particle j, and $S_{-\infty}^{(2)}$ the propagator of the particles 1 and 2 (considered as an isolated subset of the system of N particles). The limiting process $\tau\to\infty$ is possible because of the inequality $\tau_0 = r_0/v_T \ll \tau_{rel}$.

In order to obtain the collision integral I we must substitute the expression (10.4) for $g_2 = f_2 - f_1 f_1$ into the right-hand side of (10.1), with the result:

$$I(\boldsymbol{p}_1, t) = n \int d\boldsymbol{r}_2\, d\boldsymbol{p}_2 \frac{\partial \Phi_{12}}{\partial \boldsymbol{r}_1} \cdot \frac{\partial}{\partial \boldsymbol{p}_1} f_1(\boldsymbol{P}_1(-\infty), t)\, f_1(\boldsymbol{P}_2(-\infty), t) \tag{10.5}$$

The collision integral was first obtained in this form by Bogolyubov. It involves only the one-particle function f_1. Hence Eq. (10.1), with the right-hand side (10.5). is a closed equation for f_1, called the Boltzmann kinetic equation.

Originally, this equation was derived by Boltzmann by using a different method. He calculated the average number of collisions leading to an increase or a decrease of the number of particles having momenta within a range $d\boldsymbol{p}_1$ around the value $\boldsymbol{p}_1$. The collision integral was then obtained in a different form. We now show that expression (10.5) can be transformed into the classical Boltzmann form.

We multiply both sides of Eq. (10.2) by n and integrate over $\boldsymbol{r}_2$ and $\boldsymbol{p}_2$. The first terms on the left-hand side and on the right-hand side contribute the following terms to the result:

$$n \frac{\partial}{\partial t} \int d\boldsymbol{r}_2\, d\boldsymbol{p}_2 (f_2 - f_1 f_1) = N \frac{\partial}{\partial t}(f_1 - f_1) = 0 \tag{10.6}$$

We used here the normalization condition (1.14) for the function f_1 and the first equation (2.7). We note further that

$$n \int d\boldsymbol{r}_2\, d\boldsymbol{p}_2\; \frac{\partial \Phi_{12}}{\partial \boldsymbol{r}_2} \cdot \frac{\partial}{\partial \boldsymbol{p}_2} f_2 = 0 \,. \tag{10.7}$$

We thus obtain from Eq. (10.2)

$$n \int d\boldsymbol{r}_2\, d\boldsymbol{p}_2\; \frac{\partial \Phi_{12}}{\partial \boldsymbol{r}_1} \cdot \frac{\partial f_2}{\partial \boldsymbol{p}_1} = n \int d\boldsymbol{r}_2\; d\boldsymbol{p}_2 (\boldsymbol{v}_1 - \boldsymbol{v}_2) \cdot \frac{\partial f_2}{\partial \boldsymbol{r}_1} \,. \tag{10.8}$$

We note that, because of the spatial homogeneity, f_2 can be replaced by g_2 in Eq. (10.8). Then the left-hand side coincides with the collision integral (10.1) and thus

$$I(\boldsymbol{p}_1, t) = n \int d\boldsymbol{r}_2\; d\boldsymbol{p}_2 (\boldsymbol{v}_1 - \boldsymbol{v}_2) \cdot \frac{\partial}{\partial \boldsymbol{r}_1} g_2 \,. \tag{10.9}$$

This is a convenient different definition of the collision integral.

We substitute into (10.9) the expression (10.4) of the correlation function:

$$I(\boldsymbol{p}_1, t) = n \int d\boldsymbol{r}_2\; d\boldsymbol{p}_2\; (\boldsymbol{v}_1 - \boldsymbol{v}_2) \cdot \frac{\partial}{\partial \boldsymbol{r}_1} f_1(\boldsymbol{P}_1(-\infty), t)\; f_1(\boldsymbol{P}_2(-\infty), t) \,. \tag{10.10}$$

We recall that $\boldsymbol{P}_1$, $\boldsymbol{P}_2$ are the initial values of the momenta of the interacting particles 1, 2 which, at time t, have the values $\boldsymbol{p}_1$, $\boldsymbol{p}_2$. The explicit expression of these initial momenta in terms of $\boldsymbol{p}_1, \boldsymbol{r}_1, \boldsymbol{p}_2, \boldsymbol{r}_2$ is

$$\begin{aligned} \boldsymbol{P}_1(-\infty) &= \boldsymbol{P}_1(\boldsymbol{r}_1 - \boldsymbol{r}_2, \boldsymbol{p}_1, \boldsymbol{p}_2, -\infty) \\ \boldsymbol{P}_2(-\infty) &= \boldsymbol{P}_2(\boldsymbol{r}_1 - \boldsymbol{r}_2, \boldsymbol{p}_1, \boldsymbol{p}_2, -\infty) \end{aligned} \tag{10.11}$$

The momenta $\boldsymbol{P}_1, \boldsymbol{P}_2$ depend only on $\boldsymbol{r}_1 - \boldsymbol{r}_2$ and not separately on $\boldsymbol{r}_1$, $\boldsymbol{r}_2$.

In the initial state the particles are sufficiently far apart for their interaction to vanish, hence we can write the relation:

$$\frac{\boldsymbol{p}_1^2}{2m} + \frac{\boldsymbol{p}_2^2}{2m} + \Phi(|\boldsymbol{r}_1 - \boldsymbol{r}_2|) = \frac{\boldsymbol{P}_1^2}{2m} + \frac{\boldsymbol{P}_2^2}{2m} \tag{10.12}$$

which expresses the energy conservation law.

In Eq. (10.10) we may perform the integration over $\boldsymbol{r}_2$ and represent the result as a difference of two terms:

$$I = I_b - I_a \,. \tag{10.13}$$

This means that in the binary collision approximation we may distinguish two classes of collisions. One of them (collisions of type a) decrease the number of particles with momentum $\boldsymbol{p}_1$ at the point $\boldsymbol{r}_1$, whereas collisions of type b increase the particles of that kind. These collision processes are represented schematically in Fig. 1.

In the processes of type a the momenta $\boldsymbol{p}_1$, $\boldsymbol{p}_2$ play the role of 'initial' momenta, and $\boldsymbol{p}_1', \boldsymbol{p}_2'$ of 'final' momenta. Conversely, in the collisions of type b, $\boldsymbol{p}_1', \boldsymbol{p}_2'$ are initial, and $\boldsymbol{p}_1$, $\boldsymbol{p}_2$ are final momenta. In both cases, the momenta

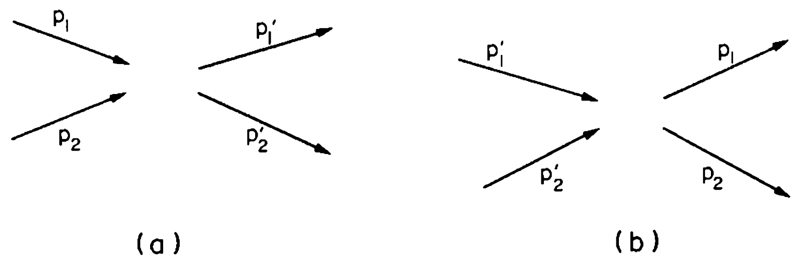

Fig. 1

$\boldsymbol{p}_1, \boldsymbol{p}_2, \boldsymbol{p}_1', \boldsymbol{p}_2'$ are related by the laws of conservation of momentum and kinetic energy:

$$\boldsymbol{p}_1 + \boldsymbol{p}_2 = \boldsymbol{p}_1' + \boldsymbol{p}_2' , \quad \frac{p_1^2}{2m} + \frac{p_2^2}{2m} = \frac{p_1'^2}{2m} + \frac{p_2'^2}{2m} . \tag{10.14}$$

We come back to Eq. (10.10) and go over to cylindrical coordinates z, ρ, ϕ. for the variable $\boldsymbol{r}_1 - \boldsymbol{r}_2$. The axis of the cylindrical coordinate frame is chosen along the relative velocity vector $\boldsymbol{v}_1 - \boldsymbol{v}_2$. Eq. (10.10) then takes the form

$$I(\boldsymbol{p}_1, t) = n \int d\boldsymbol{p}_2 \int_{-\infty}^{\infty} dz \int_0^{2\pi} d\phi \int_0^{\infty} d\rho\, \rho |\boldsymbol{v}_1 - \boldsymbol{v}_2| \frac{\partial}{\partial z} f_1(\boldsymbol{P}_1(-\infty), t)\, f_1(\boldsymbol{P}_2(-\infty), t)$$

The integration over z is easily performed:

$$I(\boldsymbol{P}_1, t) = n \int d\boldsymbol{p}_2 \int_0^{2\pi} d\phi \int_0^{\infty} d\rho\, \rho |\boldsymbol{v}_1 - \boldsymbol{v}_2| f_1(\boldsymbol{P}_1(-\infty), t)\, f_1(-\infty), t) \Big|_{z = \pm\infty} \tag{10.15}$$

For $z = -\infty$, the contribution to $I(\boldsymbol{p}_1, t)$ is negative, thus $\boldsymbol{P}_1, \boldsymbol{P}_2$ are initial values of the momenta in collisions of type a, i.e.

$$\begin{aligned} \boldsymbol{P}_1(\boldsymbol{r}_1 - \boldsymbol{r}_2, \boldsymbol{p}_1, \boldsymbol{p}_2, -\infty)\big|_{z=-\infty} &= \boldsymbol{p}_1 \\ \boldsymbol{P}_2(\boldsymbol{r}_1 - \boldsymbol{r}_2, \boldsymbol{p}_1, \boldsymbol{p}_2, -\infty)\big|_{z=-\infty} &= \boldsymbol{p}_2 \end{aligned} \tag{10.16}$$

For $z = +\infty$, the contribution to $I(\boldsymbol{p}_1, t)$ is positive, thus $\boldsymbol{P}_1, \boldsymbol{P}_2$ are initial momenta for collisions of type b, i.e.

$$\boldsymbol{P}_{1,2}(\boldsymbol{r}_1 - \boldsymbol{r}_2, \boldsymbol{p}_1, \boldsymbol{p}_2, -\infty)\big|_{z=+\infty} = \boldsymbol{p}_{1,2}' \tag{10.17}$$

As a result, Eq. (10.15) reduces to

$$I(\boldsymbol{p}_1, t) = n \int d\boldsymbol{p}_2 \int_0^{2\pi} d\phi \int_0^{\infty} d\rho\, \rho |\boldsymbol{v}_1 - \boldsymbol{v}_2| \left\{ f_1(\boldsymbol{p}_1', t)\, f_1(\boldsymbol{p}_2', t) - f_1(\boldsymbol{p}_1, t)\, f_1(\boldsymbol{p}_2, t) \right) \tag{10.18}$$

where the momenta $\boldsymbol{p}_1, \boldsymbol{p}_2, \boldsymbol{p}_1', \boldsymbol{p}_2'$ are related by the conservation laws (10.14).

The four scalar equations (10.14) are, of course, insufficient for the determination of, say, $\boldsymbol{p}_1', \boldsymbol{p}_2'$ in terms of $\boldsymbol{p}_1, \boldsymbol{p}_2$; the functions $\boldsymbol{p}_1'(\boldsymbol{p}_1, \boldsymbol{p}_2)$, $\boldsymbol{p}_2'(\boldsymbol{p}_1, \boldsymbol{p}_2)$ thus depend on two parameters, which can be chosen to be ρ, ϕ. Thus,

in (10.18)

$$\boldsymbol{p}'_{1,2} = \boldsymbol{p}'_{1,2}\,(\boldsymbol{p}_1, \boldsymbol{p}_2, \rho, \phi)\ . \tag{10.19}$$

The determination of these functions reduces to the solution of the dynamical problem of the collision of two atoms.

We recall that the quantity

$$d\sigma = \rho\, d\rho\, d\phi \tag{10.20}$$

is the differential scattering cross-section for a collision parameter ρ.

Instead of the variables ρ, ϕ we may also choose the scattering angles χ, ϕ. We must then find the dependence of the collision parameter ρ on the deflection angle χ, i.e. $\rho = \rho(\chi)$, by solving the mechanical problem. We then obtain, instead of (10.20)

$$d\sigma = \frac{\rho(\chi)}{\sin\chi}\left|\frac{d\rho}{d\chi}\right| d\Omega\ , \qquad d\Omega = \sin\chi\, d\chi\, d\phi\ . \tag{10.21}$$

Equation (10.18) is valid for an arbitrary interaction potential Φ. Let us consider as an example the case of the hard-sphere model, when

$$\Phi(r) = \begin{cases} \infty & \text{for} \quad r \leqslant r_0 \\ 0 & \text{for} \quad r > r_0 \end{cases}\ . \tag{10.22}$$

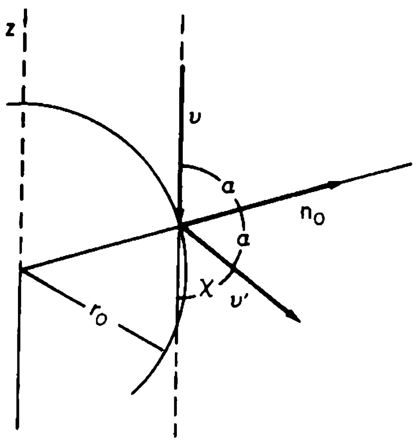

Fig. 2

We choose as parameters in Eq. (10.19) the values of the angles defining the orientation of the unit vector $\boldsymbol{n}_0$, directed along the line of centres of the particles at the instant of their collision (see Fig. 2). Eqs (10.19) become:

$$\begin{aligned} \boldsymbol{p}'_1 &= \boldsymbol{p}_1 - m(\boldsymbol{v}\cdot\boldsymbol{n}_0)\,\boldsymbol{n}_0 \\ \boldsymbol{p}'_2 &= \boldsymbol{p}_2 + m(\boldsymbol{v}\cdot\boldsymbol{n}_0)\,\boldsymbol{n}_0 \end{aligned} \tag{10.23}$$

where $\boldsymbol{v} = (\boldsymbol{p}_1 - \boldsymbol{p}_2)/m$ is the relative velocity of the colliding particles. It is easily seen that the expressions (10.23) satisfy the conservation laws (10.14). From (10.23) we find the relative velocity

$$\boldsymbol{v}' = \boldsymbol{v} - 2(\boldsymbol{v}\cdot\boldsymbol{n}_0)\,\boldsymbol{n}_0 \tag{10.24}$$

Thus

$$\boldsymbol{v}'\cdot\boldsymbol{n}_0 = -(\boldsymbol{v}\cdot\boldsymbol{n}_0); \qquad \boldsymbol{v}'\times\boldsymbol{n}_0 = \boldsymbol{v}\times\boldsymbol{n}_0 \qquad |\boldsymbol{v}'| = |\boldsymbol{v}| \tag{10.25}$$

Thus, the length of the relative velocity vector is unchanged, whereas its projection on the vector $\boldsymbol{n}_0$ changes sign.

For convenience of the calculations, we represent the atom '2' as a sphere with double radius, i.e. with radius r_0, and the atom '1' as a point (see Fig. 2).

As a result of (10.25), the angles between the vectors $\boldsymbol{v}$ and $\boldsymbol{n}_0$ on the one hand, and between $\boldsymbol{v}'$ and $\boldsymbol{n}_0$ on the other hand, are equal. We call their common value α. The deflection angle is $\chi = \pi - 2\alpha$.

The collision parameter ρ is related to α by $\rho = r_0 \sin\alpha$ and therefore, in the collision integral (10.18):

$$v\rho\, d\rho\, d\phi = v\, r_0^2 \cos\alpha \sin\alpha\, d\alpha\, d\phi = (\boldsymbol{v}\cdot\boldsymbol{n}_0)\, r_0^2\, d\Omega \tag{10.26}$$

The integration domain for α is between the limits 0 and $\pi/2$, which correspond to a half-sphere. One may also integrate over the complete sphere, but then the modulus $|\boldsymbol{v}\cdot\boldsymbol{n}_0|$ must be used instead of $\boldsymbol{v}\cdot\boldsymbol{n}_0$, and the result must be divided by 2. The final form of the collision integral for a gas of hard spheres is

$$I(\boldsymbol{p}_1, t) = \tfrac{1}{2} n\, r_0^2 \int d\boldsymbol{p}_2 \int d\Omega\, |\boldsymbol{v}\cdot\boldsymbol{n}_0| \left\{ f_1(\boldsymbol{p}_1', t)\, f_1(\boldsymbol{p}_2', t) - f_1(\boldsymbol{p}_1, t)\, f_1(\boldsymbol{p}_2, t) \right\} \tag{10.27}$$

In this expression, the momenta $\boldsymbol{p}_1', \boldsymbol{p}_2'$ are related to $\boldsymbol{p}_1, \boldsymbol{p}_2$ by (10.23).

Thus, we obtained the form of the collision integral for an arbitrary potential Φ (expression (10.18)) and for the special case of the hard-sphere model (expression (10.27)). We recall that these results are obtained under two conditions: (1) the condition of complete weakening of the initial correlations, (2) the complete neglect of the retardation of the function f_1 in the solution of (10.2).

In the next paragraph, we study the general properties of the Boltzmann collision integral. It will be shown, in particular, that this kinetic equation does not take into account the contribution of the interactions to the thermodynamic functions. It therefore describes the kinetic processes in an ideal gas.

11. PROPERTIES OF THE BOLTZMANN COLLISION INTEGRAL FOR THE IDEAL GAS

We multiply the collision integral (10.18) by an arbitrary function $\phi(\boldsymbol{p}_1)$ and integrate over $\boldsymbol{p}_1$. We introduce the notation:

$$I(t) = n \int d\boldsymbol{p}_1\ \phi(\boldsymbol{p}_1)\ I(\boldsymbol{p}_1, t). \tag{11.1}$$

We introduce into (11.1) the expression (10.18) and perform a double symmetrization: (1) We symmetrize with respect to the particles 1 and 2, i.e. we permute the variable subscripts $1 \rightleftharpoons 2$. (2). We symmetrize with respect to the momenta before and after the collision, i.e. we permute $\boldsymbol{p}_1, \boldsymbol{p}_2 \rightleftharpoons \boldsymbol{p}_1', \boldsymbol{p}_2'$. We also use the equality

$$v\rho\, d\rho\, d\phi\, d\boldsymbol{p}_1\, d\boldsymbol{p}_2 = v'\, \rho'\, d\rho'\, d\phi'\, d\boldsymbol{p}_1'\, d\boldsymbol{p}_2' \tag{11.2}$$

which results from the solution of the equations of motion of the two colliding particles and expresses the mechanical equivalence of the collisions of type a and b. After this symmetrization, $I(t)$ reduces to

$$I(t) = \tfrac{1}{4} n^2 \int d\boldsymbol{p}_1 \; d\boldsymbol{p}_2 \int_0^{2\pi} d\phi \int_0^{\infty} d\rho \; \rho \, |\boldsymbol{v}_1 - \boldsymbol{v}_2| \, [\phi(\boldsymbol{p}_1) + \Phi(\boldsymbol{p}_2) - \quad (\boldsymbol{p}_1') - \quad (\boldsymbol{p}_2')]$$

$$\cdot \left\{ f_1(\boldsymbol{p}_1', t) \, f_1(\boldsymbol{p}_2', t) - f_1(\boldsymbol{p}_1, t) \, f_1(\boldsymbol{p}_2, t) \right\} . \qquad (11.3)$$

It then follows that the integral $I(t)$ vanishes exactly, for any function $f_1(\boldsymbol{p}_1, t)$, whenever the function ϕ satisfies the equation

$$\phi(\boldsymbol{p}_1) + \phi(\boldsymbol{p}_2) = \phi(\boldsymbol{p}_1') + \phi(\boldsymbol{p}_2') \qquad (11.4)$$

We recall that $\boldsymbol{p}_1, \boldsymbol{p}_2; \boldsymbol{p}_1', \boldsymbol{p}_2'$ are, respectively, the momenta before and after the collision, thus (11.4) is satisfied for all functions ϕ which are conserved in the collision process, i.e., for $\phi(\boldsymbol{p}) = 1, \boldsymbol{p}, p^2/2m$. Hence

$$I(t) = n \int d\boldsymbol{p} \; \phi(\boldsymbol{p}) \; I(\boldsymbol{p}, t) = 0 \quad \text{for} \quad \phi = 1, \boldsymbol{p}, p^2/2m . \qquad (11.5)$$

We now consider another property of the collision integral. We choose for $\phi(\boldsymbol{p})$:

$$\phi(\boldsymbol{p}) = - k_B \ln f_1(\boldsymbol{p}_1, t)$$

and rewrite (11.3) as:

$$I(t) = -\tfrac{1}{4} k_B n^2 \int d\boldsymbol{p}_1 \int d\boldsymbol{p}_2 \int d\phi \int d\rho \; \rho \, |\boldsymbol{v}_1 - \boldsymbol{v}_2| \ln \frac{f_1(\boldsymbol{p}_1, t) f_1(\boldsymbol{p}_2, t)}{f_1(\boldsymbol{p}_1', t) f_2(\boldsymbol{p}_2', t)}$$

$$\left\{ f_1(\boldsymbol{p}_1', t) f_1(\boldsymbol{p}_2', t) - f_1(\boldsymbol{p}_1, t) f_1(\boldsymbol{p}_2, t) \right\} . \qquad (11.6)$$

It then follows that

$$I(t) \geqslant 0 \quad \text{for} \quad \phi(\boldsymbol{p}) = - k_B \ln f_1(\boldsymbol{p}, t) . \qquad (11.7)$$

The equality sign appears when the function f_1 satisfies the condition

$$f_1(\boldsymbol{p}_1, t) f_1(\boldsymbol{p}_2, t) = f_1(\boldsymbol{p}_1', t) f_1(\boldsymbol{p}_2', t) . \qquad (11.8)$$

The solution of this equation is the Maxwell distribution :

$$f_1(\boldsymbol{p}) = (2\pi m k_B T)^{-\frac{3}{2}} \exp(- p^2/2m k_B T) . \qquad (11.9)$$

Under the condition (11.8), the collision integral (10.18) vanishes, and thus (11.1) reduces to

$$\partial f_1(\boldsymbol{p}_1, t)/\partial t = 0 . \qquad (11.10)$$

Thus, the Maxwell distribution is the equilibrium solution of the Boltzmann equation for a spatially homogeneous gas. In presence of an external field, the equilibrium solution of the Boltzmann equation is the Maxwell-Boltzmann distribution.

We now use Eqs (11.5) to derive the balance equations for the mass, momentum and energy densities (Eqs (7.2), (7.9) and (7.14)). Besides the five scalar functions ρ, u_i, T, these equations involve the unknown functions (7.15). For the

Boltzmann collision integral,

$$\int d\boldsymbol{p}\ \boldsymbol{p} I = 0\,, \qquad \int d\boldsymbol{p}(\boldsymbol{p}^2/2m)\ I = 0 \tag{11.11}$$

and therefore Eqs (7.2), (7.9) and (7.14) for $\boldsymbol{F}_0 = 0$ and for a spatially homogeneous gas reduce to:

$$\frac{\partial \rho}{\partial t} = 0\,, \qquad \frac{\partial \rho \boldsymbol{u}}{\partial t} = 0\,, \qquad \frac{\partial}{\partial t}\left(\frac{3}{2}\frac{\rho}{m} k_{\mathrm{B}} T\right) = 0\,.$$

Under these conditions, the quantities

$$\rho, \quad \rho\boldsymbol{u}\,, \quad \frac{3}{2}\frac{\rho}{m} k_{\mathrm{B}} T \equiv U_{\mathrm{id}} \tag{11.12}$$

do not change with time.

The quantity U_{id} is the internal energy of the ideal gas. This shows that in the Boltzmann kinetic equation, obtained under the two assumptions discussed in section 10, the interactions do not contribute to the non-dissipative characteristics of the system. In this sense, we shall say that the Boltzmann equation (10.1) with the collision integral (10.18) is the kinetic equation for an ideal gas.

The property (11.7) is related to the growth of the entropy:

$$S(t) = -k_{\mathrm{B}} n \int d\boldsymbol{p}\ f_1(\boldsymbol{p}\,, t)\ \ln\, f_1(\boldsymbol{p}\,,t)\,. \tag{11.13}$$

Indeed, from (10.1), (10.7) and (11.13) we obtain

$$dS/dt \geqslant 0\,, \quad \text{or} \quad dH/dt \leqslant 0\,, \quad H = -S \tag{11.14}$$

(Boltzmann's H-theorem). The equality sign appears in the case of the equilibrium state.

We now show how the conservation properties (11.5) can be derived from the form (10.5) of the collision integral.

The validity of the first relation (11.5) is obvious. We examine the second relation by multiplying (10.5) by $n\boldsymbol{p}_1$ and integrating by parts over $\boldsymbol{p}_1$:

$$n \int d\boldsymbol{p}_1\ \boldsymbol{p}_1\ I(\boldsymbol{p}_1, t) = -n^2 \int d\boldsymbol{p}_1\, d\boldsymbol{p}_2\ d\boldsymbol{r}_{12} \frac{\partial \Phi_{12}}{\partial \boldsymbol{r}_{12}} f_1(\boldsymbol{P}_1(-\infty)\,, t)\ f_1(\boldsymbol{P}_2(-\infty)\,, t) \tag{11.15}$$

where $\boldsymbol{r}_{12} = \boldsymbol{r}_1 - \boldsymbol{r}_2$. As a result of the spatial homogeneity the integrand in (11.15) depends only on $\boldsymbol{r}_1 - \boldsymbol{r}_2$; thus under the substition $\boldsymbol{p}_1 \rightleftharpoons \boldsymbol{p}_2$, $\boldsymbol{r}_{12} \to -\boldsymbol{r}_{12}$ it changes sign, and therefore the expression (11.15) vanishes.

We now multiply (10.5) by $np_1^2/2m$, integrate by parts over $\boldsymbol{p}_1$ and symmetrize the result in 1 and 2. We find

$$n \int d\boldsymbol{p}_1\ (\boldsymbol{p}_1^2/2m)\ I(\boldsymbol{p}_1\,, t)$$

$$= \frac{1}{2} n^2 \int d\boldsymbol{r}_{12} d\boldsymbol{p}_1 d\boldsymbol{p}_2 (\boldsymbol{v}_2 - \boldsymbol{v}_1) \cdot \frac{\partial \Phi_{12}}{\partial \boldsymbol{r}_{12}} f_1(\boldsymbol{P}_1(-\infty)\,, t)\ f_1(\boldsymbol{P}_2(-\infty)\,, t)\,. \tag{11.16}$$

We add to this expression a time integration by writing it as

$$n\int d\boldsymbol{p}_1\,(p_1^2/2m)\,I(\boldsymbol{p}_1,t)$$
$$=\tfrac{1}{2}n^2\int_0^\infty d\tau\,V^{-1}\int dx_1\,dx_2(\boldsymbol{v}_2-\boldsymbol{v}_1)\cdot\frac{\partial\Phi_{12}}{\partial\boldsymbol{r}_{12}}\,\frac{d}{d\tau}\,f_1(\boldsymbol{P}_1(-\tau),t)\,f_1(\boldsymbol{P}_2(-\tau),t)\,. \tag{11.17}$$

We perform the change of variables $x_1, x_2\to X_1(-\tau), X_2(-\tau)$. By the Liouville theorem for the two-body problem, $dx_1\,dx_2=dX_1\,dX_2$, hence

$$n\int d\boldsymbol{p}_1\,(p_1^2/2m)\,I(\boldsymbol{p}_1,t)$$
$$=\tfrac{1}{2}n^2\int_0^\infty d\tau\,V^{-1}\int dX_1\,dX_2\left[(\boldsymbol{v}_2-\boldsymbol{v}_1)\cdot\frac{\partial\Phi_{12}}{\partial\boldsymbol{r}_{12}}\right]_\tau\frac{d}{d\tau}\,f_1(\boldsymbol{P}_1,t)\,f_1(\boldsymbol{P}_2,t) \tag{11.18}$$

The brackets $[\;]_\tau$ mean that the corresponding expression is a function of τ through the integration variables $X_1(-\tau)$, $X_2(-\tau)$.

We now use the equation of motion for an arbitrary dynamical function $A(x)$:

$$\frac{dA(x)}{dt}=[H(x)\,,A(x)] \tag{11.19}$$

For $A=f_1(\boldsymbol{P}_1(-\tau),t)\,f_1(\boldsymbol{P}_2(-\tau),t)$ we must use in (11.19) $t=-\tau$. We thus find

$$\frac{d}{d(-\tau)}\,f_1(\boldsymbol{P}_1(-\tau),t)\,f_1(\boldsymbol{P}_2(-\tau),t)$$
$$=-\left\{\left(\frac{\partial\Phi_{12}}{\partial\boldsymbol{r}_1}\cdot\frac{\partial}{\partial\boldsymbol{p}_1}+\frac{\partial\Phi_{12}}{\partial\boldsymbol{r}_2}\cdot\frac{\partial}{\partial\boldsymbol{p}_2}\right)f_1\,f_1\right\}_{X_1(-\tau),X_2(-\tau)}\,. \tag{11.20}$$

Thus, in the integrand of (11.18), the expression $(d/d\tau)f_1\,f_1$ does not depend explicitly on τ, but depends on this variable only through the integration variables $X_1(-\tau), X_2(-\tau)$. We note further that:

$$\left[(\boldsymbol{v}_1-\boldsymbol{v}_2)\cdot\frac{\partial\Phi_{12}}{\partial\boldsymbol{r}_{12}}\right]_\tau=\frac{d}{d\tau}[\Phi_{12}]_\tau\,. \tag{11.21}$$

We now substitute (11.20), (11.21) into (11.18) and note that by integration over τ only the contribution from $\tau=0$ remains; however, $X_{1,2}(-\tau)\big|_{\tau=0}=x_{1,2}$. Hence

$$n\int d\boldsymbol{p}_1\,(p_1^2/2m)\,I(\boldsymbol{p}_1,t)$$
$$=\tfrac{1}{2}n^2\int dx_1\,dx_2\,V^{-1}\,\Phi_{12}\left(\frac{\partial\Phi_{12}}{\partial\boldsymbol{r}_1}\cdot\frac{\partial}{\partial\boldsymbol{p}_1}+\frac{\partial\Phi_{12}}{\partial\boldsymbol{r}_2}\cdot\frac{\partial}{\partial\boldsymbol{p}_2}\right)f_1(\boldsymbol{p}_1,t)\,f_1(\boldsymbol{p}_2,t)=0. \tag{11.22}$$

We have thus proven the third relation (11.5).

This proof of Eq. (11.22) will be used in section 12 for the study of the conservation relations for a nonideal gas.

12. THE BOLTZMANN KINETIC EQUATION FOR A SPATIALLY HOMOGENEOUS NONIDEAL GAS

We come back to Eq. (10.2) and consider its solution by retaining only the assumption of the complete weakening of the initial correlations. Thus, we do not neglect the time retardation of the functions f_1. Instead of (10.4) we then obtain the following solution:

$$f_2(x_1,x_2,t) = f_1(P_1(-\tau),t-\tau)\ f_1(P_2(-\tau),t-\tau) + \int_0^\tau d\tau' \left[\frac{\partial}{\partial t} f_1(p_1,t)\ f_1(p_2,t)\right]_{P_{1,2}(-\tau'),\,t-\tau'} . \quad (12.1)$$

In first approximation with respect to the retardation (i.e. to first order in $\tau_0(\partial/\partial t)\sim\tau_0/\tau_{\mathbf{rel}}$) we obtain

$$f_2(x_1,x_2,t) = \left(1-\tau\frac{\partial}{\partial t}\right) f_1(P_1(-\tau),t)\ f_1(P_2(-\tau),t) + \int_0^\tau d\tau' \left[\frac{\partial}{\partial t} f_1 f_1\right]_{P_{1,2}(-\tau'),\,t} . \quad (12.2)$$

In the second term we integrate by parts over τ':

$$f_2(x_1,x_2,t) = f_1(P_1(-\tau),t)\ f_1(P_2(-\tau),t) - \frac{\partial}{\partial t}\int_0^\tau d\tau'\ \tau' \frac{\partial}{\partial\tau'} f_1(P_1(-\tau'),t)\ f_1(P_2(-\tau'),t) . \quad (12.3)$$

We may now go to the limit $\tau\to\infty$. We thus obtain the solution of (10.2), including the contribution of the retardation, to first approximation in the small parameter $\tau_0/\tau_{\mathbf{rel}}$:

$$f_2(x_1,x_2,t) = f_1(P_1(-\infty),t)\ f_1(P_2(-\infty),t) - \frac{\partial}{\partial t}\int_0^\infty d\tau\ \tau\frac{\partial}{\partial\tau} f_1(P_1(-\tau),t)\ f_1(P_2(-\tau),t) . \quad (12.4)$$

The first term is the same as in (10.4), whereas the second term takes account of the retardation.

We substitute (12.4) for $g_2=f_2-f_1 f_1$ into the right-hand side of (10.1) and represent the resulting collision integral as a sum of two terms:†

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† The nonideality correction $I_{(2)}$ was derived, to all orders in the retardation and in the interaction strength, by Prigogine and Résibois [71]. They expressed this correction term as an operator Ω, acting on the 'ideal' collision operator Ψ. (They did not use the terminology 'ideal' and 'nonideal'.) It is not difficult to check the equivalence, to first order in the retardation and in the density parameter, between their result and Eq. (12.6). See also [6, 67, 68]. (Translator.)

$$I(\boldsymbol{p}_1, t) = I_{(1)}(\boldsymbol{p}_1, t) + I_{(2)}(\boldsymbol{p}_1, t) \tag{12.5}$$

The first term is the same as (10.5), whereas the second term is

$$I_{(2)}(\boldsymbol{p}_1, t)$$
$$= - n \frac{\partial}{\partial t} \int_0^\infty d\tau \int d\boldsymbol{r}_2 \, d\boldsymbol{p}_2 \frac{\partial \Phi_{12}}{\partial \boldsymbol{r}_1} \cdot \frac{\partial}{\partial \boldsymbol{p}_1} \tau \frac{\partial}{\partial \tau} f_1(\boldsymbol{P}_1(-\tau), t) \, f_1(\boldsymbol{P}_2(-\tau), t) \, . \tag{12.6}$$

The integral $I_{(1)}$ possesses the properties (11.5), i.e.

$$n \int d\boldsymbol{p}_1 \; \phi(\boldsymbol{p}_1) \; I_{(1)}(\boldsymbol{p}_1, t) = 0 \quad \text{for} \quad \phi = 1, \boldsymbol{p}_1, \boldsymbol{p}_1^2/2m \; . \tag{12.7}$$

We now consider the corresponding relations for the integral $I_{(2)}$. From (12.6) it is obvious that

$$\int d\boldsymbol{p}_1 \; I_2(\boldsymbol{p}_1, t) = 0 \; .$$

We multiply (12.6) by $n\boldsymbol{p}_1$ and integrate by parts over $\boldsymbol{p}_1$:

$$n \int d\boldsymbol{p}_1 \; \boldsymbol{p}_1 I_{(2)}$$
$$= n \frac{\partial}{\partial t} \int_0^\infty d\tau \int d\boldsymbol{p}_1 \, d\boldsymbol{p}_2 \; d\boldsymbol{r}_{12} \, \tau \frac{\partial \Phi_{12}}{\partial \boldsymbol{r}_{12}} \frac{\partial}{\partial \tau} f_1(\boldsymbol{P}_1(-\tau), t) \;\; f_1(\boldsymbol{P}_2(-\tau), t). \tag{12.8}$$

Because of the spatial homogeneity, as before in (11.15), the integrand depends only on $(\boldsymbol{r}_1 - \boldsymbol{r}_2)$ and changes sign under the substitution $\boldsymbol{p}_1 \to \boldsymbol{p}_2$, $\boldsymbol{r}_{12} \to -\boldsymbol{r}_{12}$; thus the expression (12.8) vanishes.

Thus

$$n \int d\boldsymbol{p}_1 \; \phi(\boldsymbol{p}_1) \; I_{(2)}(\boldsymbol{p}_1, t) = 0 \qquad \text{for} \quad \phi = 1, \boldsymbol{p}_1 \; . \tag{12.9}$$

We now show that, in contrast to (11.5), the equality (12.9) does not hold for $\phi = p_1^2/2m$. Multiplying (12.6) by the latter factor, integrating by parts over $\boldsymbol{p}_1$ and symmetrizing the result in 1 and 2 we find

$$n \int d\boldsymbol{p}_1 \, (p_1^2/2m) \; I_{(2)}$$
$$= \tfrac{1}{2} n \frac{\partial}{\partial t} \int_0^\infty d\tau \; V^{-1} \int dx_1 \, dx_2 (\boldsymbol{v}_1 - \boldsymbol{v}_2) \cdot \frac{\partial \Phi_{12}}{\partial \boldsymbol{r}_{12}} \tau \frac{\partial}{\partial \tau} f_1(\boldsymbol{P}_1(-\tau), t) \; f_1(\boldsymbol{P}_2(-\tau), t) \, . \tag{12.10}$$

This expression corresponds to (11.17). As for the latter expression, we change the integration variables: $x_1, x_2 \to X_1(-\tau), X_2(-\tau)$, and use Eq. (11.21):

$$n \int d\boldsymbol{p}_1 \; (p_1^2/2m) I_{(2)}$$
$$= \tfrac{1}{2} n^2 \frac{\partial}{\partial t} \int_0^\infty d\tau \; V^{-1} \int dX_1 \, dX_2 \frac{\partial}{\partial \tau} [\Phi_{12}]_\tau \, \tau \, \frac{\partial}{\partial \tau} f_1(\boldsymbol{P}_1(-\tau), t) \;\; f_1(\boldsymbol{P}_2(-\tau), t). \tag{12.11}$$

We perform in (12.11) an integration by parts over τ, recalling that the integrand in (12.11) does not depend explicitly on τ; thus:

$$\int_0^\infty d\tau\ \tau \frac{\partial}{\partial \tau} [\Phi_{12}]_\tau = -\int_0^\infty d\tau\ [\Phi_{12}]_\tau . \tag{12.12}$$

After reverting to the old integration variables, we thus obtain:

$$n \int d\boldsymbol{p}_1 (p_1^2/2m)\, I_{(2)}$$
$$= -\tfrac{1}{2} n^2 \frac{\partial}{\partial t} \int_0^\infty d\tau\, V^{-1} \int dx_1\, dx_2\, \Phi_{12} \frac{\partial}{\partial \tau} f_1(\boldsymbol{P}_1(-\tau), t)\ f_1(\boldsymbol{P}_2(-\tau), t) . \tag{12.13}$$

The integration over τ can now be performed. Noting that, for a spatially homogeneous state:

$$\frac{\partial}{\partial t} \int d\boldsymbol{p}_1\, d\boldsymbol{p}_2\, f_1(\boldsymbol{p}_1, t)\ f_1(\boldsymbol{p}_2, t) = 0 , \tag{12.14}$$

we obtain our final result:

$$n \int d\boldsymbol{p}_1 (p_1^2/2m)\, I_{(2)}(\boldsymbol{p}_1, t)$$
$$= -\frac{\partial}{\partial t} \tfrac{1}{2} n^2 V^{-1} \int dx_1\, dx_2\, \Phi_{12}\, f_1(\boldsymbol{P}_1(-\infty), t)\ f_1(\boldsymbol{P}_2(-\infty), t) . \tag{12.15}$$

We consider again the energy balance equation. From (7.14), (12.7) and (12.15) we obtain:

$$\frac{\partial}{\partial t} \Big\{ n \int d\boldsymbol{p}_1\, \frac{p_1^2}{2m} f_1(\boldsymbol{p}_1, t) +$$
$$+ \frac{n^2}{2} \int d\boldsymbol{r}_{12}\, d\boldsymbol{p}_1\, d\boldsymbol{p}_2\, \Phi_{12}\, f_1(\boldsymbol{P}_1(-\infty), t)\ f_1(\boldsymbol{P}_2(-\tau), t) \Big\} = 0 . \tag{12.16}$$

Thus, by taking the retardation effect into account, we find that the conserved quantity is the sum of the kinetic and potential energy of the colliding particles. The internal energy density must now be defined as:

$$U = n \int d\boldsymbol{p}_1\, \frac{p_1^2}{2m} f_1(\boldsymbol{p}_1, t)$$
$$+ \tfrac{1}{2} n^2 \int d\boldsymbol{r}_{12}\, d\boldsymbol{p}_1\, d\boldsymbol{p}_2\, \Phi_{12}\, f_1(\boldsymbol{P}_1(-\infty), t)\ f_1(\boldsymbol{P}_2(-\infty), t) . \tag{12.17}$$

In the equilibrium state we have:

$$f_1(\boldsymbol{P}_1(-\infty), t)\ f_1(\boldsymbol{P}_2(-\infty), t)$$
$$= (2\pi m\, k_{\mathrm{B}} T)^{-3} \exp\left[- \frac{(p_1^2/2m) + (p_2^2/2m) + \Phi_{12}}{k_{\mathrm{B}} T} \right] . \tag{12.18}$$

We used here Eq. (10.12). Using Eq. (12.18) we get for the internal energy density:

$$U = \frac{3}{2} n k_B T + \frac{1}{2} n^2 \int d\boldsymbol{r}_{12}\, \Phi_{12} \exp\left(-\frac{\Phi_{12}}{k_B T}\right) . \tag{12.19}$$

This expression coincides with the result (8.12) of the equilibrium theory, obtained in the first approximation in the density parameter.

13. THE COLLISION INTEGRAL IN THE WEAK COUPLING APPROXIMATION. THE LANDAU KINETIC EQUATION

In the weak coupling approximation we use, instead of Eq. (10.2), the simpler equation (3.4). In the case $\boldsymbol{F}_0 = 0$, and for a spatially homogeneous state it reduces to

$$\left(\frac{\partial}{\partial t} + \boldsymbol{v}_1 \cdot \frac{\partial}{\partial \boldsymbol{r}_1} + \boldsymbol{v}_2 \cdot \frac{\partial}{\partial \boldsymbol{r}_2}\right) g_2(x_1, x_2, t)$$

$$= \left(\frac{\partial \Phi_{12}}{\partial \boldsymbol{r}_1} \cdot \frac{\partial}{\partial \boldsymbol{p}_1} + \frac{\partial \Phi_{12}}{\partial \boldsymbol{r}_2} \cdot \frac{\partial}{\partial \boldsymbol{p}_2}\right) f_1(\boldsymbol{p}_1, t)\, f_1(\boldsymbol{p}_2, t) \tag{13.1}$$

By using the condition of complete weakening of the initial correlations, we conserve only the solution of the inhomogeneous equation (13.1). Taking account of the retardation effect, the solution is written as

$$g_2(x_1, x_2, t) = \int_0^\infty d\tau \left[\frac{\partial}{\partial \boldsymbol{r}_1} \Phi_{12}(|\boldsymbol{r}_1 - \boldsymbol{r}_2 - (\boldsymbol{v}_1 - \boldsymbol{v}_2)\tau|)\right] \cdot$$

$$\cdot \left(\frac{\partial}{\partial \boldsymbol{p}_1} - \frac{\partial}{\partial \boldsymbol{p}_2}\right) f_1(\boldsymbol{p}_1, t-\tau)\, f_1(\boldsymbol{p}_2, t-\tau) \tag{13.2}$$

This expression defines the correlation function g_2 in the weak coupling approximation. To first order in the retardation, it reduces to :

$$g_2(x_1, x_2, t) = \int_0^\infty d\tau \left[\frac{\partial}{\partial \boldsymbol{r}_1} \Phi_{12}(|\boldsymbol{r}_1 - \boldsymbol{r}_2 - (\boldsymbol{v}_1 - \boldsymbol{v}_2)\tau|)\right] \cdot$$

$$\cdot \left(1 - \tau \frac{\partial}{\partial t}\right)\left(\frac{\partial}{\partial \boldsymbol{p}_1} - \frac{\partial}{\partial \boldsymbol{p}_2}\right) f_1(\boldsymbol{p}_1, t)\, f_1(\boldsymbol{p}_2, t) . \tag{13.3}$$

We now show how this expression can be derived directly from the solution (12.4). Using the equations of motion of the particles 1 and 2 :

$$\frac{d\boldsymbol{p}_{1,2}}{dt} = -\frac{\partial \Phi_{12}}{\partial \boldsymbol{r}_{1,2}} \tag{13.4}$$

we express $\boldsymbol{p}_1$, $\boldsymbol{p}_2$, in terms of $\boldsymbol{P}_1(-\tau)$, $\boldsymbol{P}_2(-\tau)$:

$$\boldsymbol{p}_{1,2} = \boldsymbol{P}_{1,2}(-\tau) - \int_{t-\tau}^{t} dt' \left[\frac{\partial \Phi_{12}}{\partial \boldsymbol{r}_{1,2}}\right]_{t'}$$

$$= \boldsymbol{P}_{1,2}(-\tau) - \int_0^{\tau} d\tau' \left[\frac{\partial \Phi_{12}}{\partial \boldsymbol{r}_{1,2}}\right]_{t-\tau'} \tag{13.5}$$

In the weak coupling approximation, it is sufficient to calculate the functions $\boldsymbol{r}_{1,2}(t-\tau)$ to zeroth order in the interactions:

$$\boldsymbol{r}_{1,2}(t-\tau)=\boldsymbol{r}_{1,2}(t)-\boldsymbol{v}_{1,2}\,\tau\ . \tag{13.6}$$

As a result, we obtain from (13.5):

$$\boldsymbol{P}_{1,2}(-\tau)=\boldsymbol{p}_{1,2}+\int_0^\tau d\tau'\ \frac{\partial}{\partial \boldsymbol{r}_{1,2}}\ \Phi_{12}(|\boldsymbol{r}_1-\boldsymbol{r}_2-(\boldsymbol{v}_1-\boldsymbol{v}_2)\tau|)\ . \tag{13.7}$$

We substitute this expression into (12.4) and expand the result: we obtain Eq. (13.3).

We now derive the expression of the collision integral. From (13.2), (10.1) we obtain the expression of the collision integral in the weak coupling approximation, but without assuming the smallness of the retardation:

$$I(\boldsymbol{p}_1,t)=n\int_0^\infty d\tau\int dx_2\ \frac{\partial\Phi_{12}}{\partial r_{1i}}\,\frac{\partial}{\partial p_{1i}}\ \cdot$$
$$\cdot\left(\frac{\partial}{\partial r_{1j}}\,\Phi_{12}(|\boldsymbol{r}_1-\boldsymbol{r}_2-(\boldsymbol{v}_1-\boldsymbol{v}_2)\tau|)\right)\left(\frac{\partial}{\partial p_{1j}}-\frac{\partial}{\partial p_{2j}}\right)f_1(\boldsymbol{p}_1,t-\tau)\ f_1(\boldsymbol{p}_2,t-\tau)\ . \tag{13.8}$$

We may also obtain a different form, by introducing the Fourier transform of the interaction potential:

$$\Phi(\boldsymbol{r})=(2\pi)^{-3}\int d\boldsymbol{k}\ e^{i\boldsymbol{k}\cdot\boldsymbol{r}}\ \Phi(\boldsymbol{k})\ . \tag{13.9}$$

Substituting the result into (13.8) and integrating over $\boldsymbol{r}_2$, we obtain:

$$I(\boldsymbol{p}_1,t)=\frac{n}{(2\pi)^3}\,\frac{\partial}{\partial p_{1i}}\int_0^\infty d\tau\int d\boldsymbol{p}_2\int d\boldsymbol{k}\ k_i\,k_j\ \Phi^2(\boldsymbol{k})\ \cdot$$
$$\cdot\ \exp\left[-i\boldsymbol{k}\cdot(\boldsymbol{v}_1-\boldsymbol{v}_2)\tau\right]\left(\frac{\partial}{\partial p_{1j}}-\frac{\partial}{\partial p_{2j}}\right)f_1(\boldsymbol{p}_1,t-\tau)\ f_1(p_2,t-\tau)\ . \tag{13.10}$$

To first order in the retardation effect, we represent again the collision integral as a sum of two terms:

$$I=I_{(1)}+I_{(2)}$$

with

$$I_{(1)}=\frac{n}{8\pi^2}\,\frac{\partial}{\partial p_{1i}}\int d\boldsymbol{k}\ d\boldsymbol{p}_2\ k_i\,k_j\ \Phi^2(\boldsymbol{k})\,\delta\,(\boldsymbol{k}\cdot\boldsymbol{v}_1-\boldsymbol{k}\cdot\boldsymbol{v}_2)\left(\frac{\partial}{\partial p_{1j}}-\frac{\partial}{\partial p_{2j}}\right)\cdot$$
$$\cdot\ f_1(\boldsymbol{p}_1,t)\ f_1(\boldsymbol{p}_2,t)\ , \tag{13.12}$$

$$I_{(2)}=-\frac{n}{(2\pi)^3}\,\frac{\partial}{\partial t}\,\frac{\partial}{\partial p_{1i}}\int d\boldsymbol{k}\ d\boldsymbol{p}_2\ k_i\,k_j\ \Phi^2(\boldsymbol{k})\left[\int_0^\infty d\tau\ \tau\cos(\boldsymbol{k}\cdot\boldsymbol{v}_1\,\tau-\boldsymbol{k}\cdot\boldsymbol{v}_2\,\tau)\right]\cdot$$
$$\cdot\left(\frac{\partial}{\partial p_{1j}}-\frac{\partial}{\partial p_{2j}}\right)f_1(\boldsymbol{p}_1,t)\ f_1(\boldsymbol{p}_2,t)\ . \tag{13.13}$$

Expression (13.12), which defines the collision integral in absence of retardation effects, in the weak coupling approximation, is called the Landau collision integral. We will see that this approximation is widely used for systems of charged particles, interacting through Coulomb forces.

The collision integrals (13.12), (13.13) possess the same properties as those of the binary collision approximation quantities considered previously.

Till now, we did not yet investigate the role of the additional collision integrals $I_{(2)}$ in the formulation of Boltzmann's H-theorem. This will be the object of the next section.

14. BOLTZMANN'S H-THEOREM FOR THE NONIDEAL GAS

In section 8 we presented the calculation of the thermodynamic functions to zeroth and first order in the density parameter. The zeroth approximation corresponds to the ideal gas, whereas the first order represents the contribution of the interactions to the thermodynamic quantities. In particular, the entropy S was calculated, with the result:

$$S = S_{\mathbf{id}} + \Delta S \tag{14.1}$$

where $S_{\mathbf{id}}$ is the entropy density of the ideal gas, and ΔS is the contribution of the interactions: to first order in the density, the latter is given by (8.17).

In section 11, by using the Boltzmann kinetic equation, we found an expression for the entropy of a non-equilibrium state:

$$S_{\mathbf{B}}(t) = - k_{\mathbf{B}} n \int d\boldsymbol{p}\; f_1(\boldsymbol{p},t)\; \ln f_1(\boldsymbol{p},t)\;. \tag{14.2}$$

In equilibrium, f_1 is Maxwellian, and (14.2) reduces to $S_{\mathbf{B}} = S_{\mathbf{id}}$. This shows that (14.2) does not contain the terms of first order in the interactions which are responsible, in equilibrium, for the correction (8.17).

This means that (14.2) does not account for the contribution of the interaction. Indeed, the distribution functions $f_1, f_2, \ldots$ can be expanded in powers of the density parameter. In the binary collision approximation, it is sufficient to keep the first two terms; thus, for instance:

$$f_1 = f_1^0 + \varepsilon f_1^1 \tag{14.3}$$

where f_1^0 is the ideal gas distribution function, and f_1^1 the contribution of the interactions. The latter is defined in terms of the correlation function g_2. In the binary collision approximation, we obviously get

$$f_1(\boldsymbol{p}_1,t) = C\left\{f_1^0 + n\,V^{-1} \int d\boldsymbol{r}_1\; d\boldsymbol{r}_2\, d\boldsymbol{p}_2\;\; g_2(x_1,x_2,t)\right\}. \tag{14.4}$$

The constant C is defined by requiring the normalization (1.14) of f_1. We only need terms of zeroth and first order in the density, thus

$$f_1(\boldsymbol{p}_1, t) = f_1^0(\boldsymbol{p}_1, t) + n\, V^{-1} \int d\boldsymbol{r}_1\, d\boldsymbol{r}_2 \left\{ \int d\boldsymbol{p}_2\, g_2(\boldsymbol{r}_1, \boldsymbol{r}_2, \boldsymbol{p}_1, \boldsymbol{p}_2, t) \cdot \right.$$
$$\left. \cdot - \int d\boldsymbol{p}_1'\, d\boldsymbol{p}_2\;\; g_2(\boldsymbol{r}_1, \boldsymbol{r}_2, \boldsymbol{p}_1', \boldsymbol{p}_2, t)\;\; f_1^0(\boldsymbol{p}_1, t) \right\}. \tag{14.5}$$

In equilibrium, the term proportional to n in this equation vanishes and thus

$$f_1(\boldsymbol{p}_1) = f_1^0(\boldsymbol{p}_1). \tag{14.6}$$

Thus, away from equilibrium, Eq. (14.2) contains the contribution of the interactions to the one-particle function. However, as we saw, this correction is not complete.

We now derive an expression for the entropy which, to first order in the density, contains the complete contribution of the interactions. We also show that, within the framework of the binary collision approximation, this expression satisfies the Boltzmann H-theorem for nonideal gases [37].

We represent the entropy in the form:

$$S = S_{\mathrm{B}} + \Delta S \tag{14.7}$$

where ΔS is a correction to the entropy, due to the correlations.

In order to define ΔS we proceed as follows. We rewrite the Boltzmann contribution in the form:

$$-2\,\tfrac{1}{2}\,k_{\mathrm{B}}\, V^{-2} \int dx_1\, dx_2\, f_1\, f_1\;\; \ln\,(f_1\, f_1) = -\; k_{\mathrm{B}}\, V^{-2} \int dx_1\, dx_2\, f_2\, \ln\,(f_1\, f_1)$$

where we used the normalization (1.14) and Eq. (2.8).

The entropy of a single pair of interacting particles is taken to be:

$$-\, k_{\mathrm{B}}\, V^{-2} \int dx_1\, dx_2\, f_2\;\; \ln f_2 \,.$$

As the number of pairs is $N(N-1)/2$, the two expressions can be combined to yield an expression for the entropy density due to the correlations [36, 37, 69]:

$$\Delta S = -\tfrac{1}{2}\, k_{\mathrm{B}}\, n^2\, V^{-1} \int dx_1\, dx_2\, f_2\;\; \ln \frac{f_2}{f_1\, f_1}\,. \tag{14.8}$$

We now consider this expression in equilibrium. The momentum integrations can then be easily performed and we are left with

$$\Delta S = -\tfrac{1}{2}\, k_{\mathrm{B}}\, n^2\, V^{-1} \int d\boldsymbol{r}_1\;\; d\boldsymbol{r}_2\, f_2(\boldsymbol{r}_1, \boldsymbol{r}_2)\;\; \ln f_2(\boldsymbol{r}_1, \boldsymbol{r}_2), \tag{14.9}$$

where

$$f_2(\boldsymbol{r}_1, \boldsymbol{r}_2) = C\, \exp(-\Phi_{12}/k_{\mathrm{B}}T). \tag{14.10}$$

The normalization constant is

$$C = \left\{ V^{-2} \int d\boldsymbol{r}_1\; d\boldsymbol{r}_2\;\; \exp\,(-\,\Phi_{12}/k_{\mathrm{B}}T) \right\}^{-1} \sim 1 - \frac{r_{\mathrm{corr}}}{V} \tag{14.11}$$

where r_{corr} is the correlation radius.

For $V \to \infty$, $C \to 1$, but in (14.9) this constant can be set equal to one only in the second factor of the integrand. Indeed,

$$-\frac{1}{2} k_{\mathrm{B}} n^2 V^{-1} \int d\boldsymbol{r}_1 \; d\boldsymbol{r}_2 \; f_2 \;\; \ln C = -\frac{1}{2} k_{\mathrm{B}} n N \;\ln\; C$$

$$= \frac{1}{2} k_{\mathrm{B}} n^2 V^{-1} \int d\boldsymbol{r}_1 \; d\boldsymbol{r}_2 \left[\exp\left(-\Phi_{12}/k_{\mathrm{B}} T\right) - 1\right]$$

which is of the same order as the other terms in (14.9). Using this result we obtain:

$$\Delta S \doteq \frac{1}{2} k_{\mathrm{B}} n^2 \int d\boldsymbol{r} \left\{ \frac{\Phi}{k_{\mathrm{B}} T} \exp\left(-\Phi/k_{\mathrm{B}} T\right) + \left[\exp\left(-\Phi/k_{\mathrm{B}} T\right) - 1\right] \right\} \tag{14.12}$$

where $\boldsymbol{r} = \boldsymbol{r}_1 - \boldsymbol{r}_2$. This expression is the same as (8.17). The first term represents the internal energy, and the second, the free energy.

For the model of weakly attracting hard spheres:

$$\Delta S = -k_{\mathrm{B}} n^2 b - \pi k_{\mathrm{B}} n^2 \int_{r_0}^{\infty} dr\, r^2 \left[\Phi^2(r)/(k_{\mathrm{B}} T)^2\right]$$

where $b = \frac{2}{3}\pi r_0^3$ is the van der Waals constant.

The general expression (14.8), away from equilibrium, can be written in the form of the sum of two terms, one of which reduces in equilibrium to the internal energy and the second to the free energy. To this purpose we represent the functions f_1, f_2 as

$$\begin{aligned} f_1 &= \left(V^{-1} \int dx_1 \, F_1\right)^{-1} F_1 \\ f_1 &= \left(V^{-2} \int dx_1 \, dx_2 \, F_2\right)^{-1} F_2 \\ F_2 &= F_1 F_1 + G_2 \;. \end{aligned} \tag{14.13}$$

In equilibrium

$$\begin{aligned} F_2 &= \exp\left(-\Phi/k_{\mathrm{B}} T\right) \\ G_2 &= \exp\left(-\Phi/k_{\mathrm{B}} T\right) - 1 \;. \end{aligned} \tag{14.14}$$

Equation (14.14) differs from (14.10) only in that here $C = 1$, i.e., the normalization factor does not contain the contribution from the correlations. Noting that

$$V^{-1} \int dx_1 \, F_1 = 1 + O(n)$$

and neglecting terms of order n^3, we obtain

$$\frac{1}{2}\,k_B\,n^2\,V^{-1}\int dx_1\,dx_2\,f_2\;\ln\frac{V^{-1}\int dx_1\,dx_2\,F_2}{V^{-1}\int dx_1\,F_1\,V^{-1}\int dx_2\,F_1}$$

$$=\frac{1}{2}\,k_B\,n\,N\;\ln\left\{1+\frac{V^{-2}\int dx_1\,dx_2\,G_2}{V^{-1}\int dx_1\,F_1\,V^{-1}\int dx_2\,F_1}\right\}$$

$$=\frac{1}{2}\,k_B\,n^2\,V^{-1}\int dx_1\,dx_2\,G_2\,.$$

As a result, (14.8) reduces to [37]

$$\Delta S=-\frac{1}{2}\,k_B\,n^2\,V^{-1}\int dx_1\,dx_2\left[F_2\,\ln\frac{F_2}{F_1F_1}-G_2\right]. \tag{14.15}$$

In equilibrium, by using (14.14), we recover again (14.12). We see that the first term is the non-equilibrium analogue of the contribution to the internal energy, and the second term corresponds to the free energy.

We now derive the H-function for the nonideal gas [37]. From (14.8), (14.15) we see that the correlations enter ΔS through f_2 and $\ln f_2$. In order to calculate the contribution of the logarithm of the correlations, we proceed as follows. From the kinetic equation we derive an equation for $f_1 f_1$. We multiply the latter by

$$V^{-2}\,dx_1\,dx_2\left\{-\frac{1}{2}\,k_B\,n\;\ln(f_1\,f_1)-\frac{1}{2}\,k_B\,n^2\,V\,\ln\frac{f_2}{f_1\,f_1}\right\}$$

and integrate over x_1, x_2. Neglecting terms of order n^3 on the right-hand side, we obtain

$$-k_B\,n\,V^{-1}\int dx_1\,\frac{\partial f_1}{\partial t}\,\ln f_1-\frac{1}{2}\,k_B\,n^2\,V^{-1}\int dx_1\,dx_2\,\frac{\partial f_1\,f_1}{\partial t}\,\ln\frac{f_2}{f_1\,f_1}$$
$$=-k_B\,n\,V^{-1}\int dx_1\,I_{(1)}\,\ln f_1-k_B\,n\,V^{-1}\int dx_1\,I_{(2)}\,\ln f_1\equiv I_1+I_2\,. \tag{14.16}$$

The integral $I_{(1)}$ is the same as the Boltzmann collision integral, thus:

$$I_1=-k_B\,n\,V^{-1}\int dx_1\,I_{(1)}\,\ln f_1\geqslant 0\,. \tag{14.17}$$

Using (12.6), integrating by parts over $\boldsymbol{p}_1$ and symmetrizing, we obtain

$$I_2=-\frac{1}{2}\,k_B\,n^2\,V^{-1}\int_0^\infty d\tau\int dx_1\,dx_2\left(\frac{\partial\Phi_{12}}{\partial\boldsymbol{r}_1}\cdot\frac{\partial}{\partial\boldsymbol{p}_1}+\frac{\partial\Phi_{12}}{\partial\boldsymbol{r}_2}\cdot\frac{\partial}{\partial\boldsymbol{p}_2}\right)\ln(f_1\,f_1)\,\cdot$$
$$\cdot\;\tau\,\frac{\partial}{\partial\tau}\,\frac{\partial}{\partial t}\,S_{-\tau}(1,2)\,f_1(\boldsymbol{p}_1,t)\,f_1(\boldsymbol{p}_2,t)\;.$$

We now use a transformation similar to that used in sections 11 and 12 for the proof of the energy conservation. We use new integration variables $x_{1,2}\to X_{1,2}(-\tau)$ in I_2 and note that

$$- \left[\left(\frac{\partial \Phi_{12}}{\partial \boldsymbol{r}_1} \cdot \frac{\partial}{\partial \boldsymbol{p}_1} + \frac{\partial \Phi_{12}}{\partial \boldsymbol{r}_2} \cdot \frac{\partial}{\partial \boldsymbol{p}_2} \right) \ln (f_1 f_1) \right]_{X_{1,2}(-\tau)}$$

$$= \frac{d}{d(-\tau)} \left[\ln (f_1 \; f_1) \right]_{X_{1,2}(-\tau)} .$$

We integrate by parts over τ and revert to the old variables $x_{1,2}$:

$$I_2 = -\tfrac{1}{2} k_B n^2 V^{-1} \int_0^\infty d\tau \int dx_1 \, dx_2 \ln (f_1 f_1) \frac{\partial}{\partial t} \frac{\partial}{\partial \tau} S_{-\tau}(1,2) f_1(\boldsymbol{p}_1, t) f_1(\boldsymbol{p}_2, t) \; .$$

We integrate over τ and note that, to zeroth order in the retardation,

$$g_2(x_1, x_2) = f_1(\boldsymbol{P}_1(-\tau), t) f_1(\boldsymbol{P}_2(-\tau), t) - f_1(\boldsymbol{p}_1, t) f_1(\boldsymbol{p}_2, t) \; .$$

We thus obtain:

$$I_2(t) = -\tfrac{1}{2} k_B n^2 V^{-1} \int dx_1 \, dx_2 \frac{\partial g_2}{\partial t} \ln(f_1 f_1) \; . \tag{14.18}$$

We now use Eq. (10.2) in order to put (14.18) into a more convenient form. Eq. (10.2) is multiplied by $\ln f_2$ and integrated over x_1, x_2:

$$V^{-1} \int dx_1 \, dx_2 (\ln f_2) \frac{\partial g_2}{\partial t} = 0 \; . \tag{14.19}$$

From Eqs (14.18) and (14.19) we obtain

$$I_2(t) = \tfrac{1}{2} k_B n^2 V^{-1} \int dx_1 \, dx_2 \frac{\partial g_2}{\partial t} \ln \frac{f_2}{f_1 f_1} \; . \tag{14.20}$$

As a result, we obtain from (14.16), (14.17), (14.19) the inequality

$$- k_B n V^{-1} \int dx_1 \frac{\partial f_1}{\partial t} \ln f_1 - \tfrac{1}{2} k_B n^2 V^{-1} \int dx_1 \, dx_2 \frac{\partial f_2}{\partial t} \ln \frac{f_2}{f_1 f_1} = I_1(t) \geqslant 0. \tag{14.21}$$

We note the identities

$$\begin{aligned} &V^{-1} \int dx_1 \; f_1 \frac{\partial}{\partial t} \ln f_1 = 0 \\ &V^{-1} \int dx_1 \, dx_2 \, f_2 \frac{\partial}{\partial t} \ln \frac{f_2}{f_1 f_1} = - V^{-1} \int dx_1 \, dx_2 \frac{\partial g_2}{\partial t} = 0 . \end{aligned} \tag{14.22}$$

The latter relation follows from (10.2). We may thus write the symbol $\partial/\partial t$ in front of the integral sign on the left-hand side of (14.21) and get

$$\frac{\partial S}{\partial t} = - \frac{\partial H}{\partial t} = I_1(t) \geqslant 0 \tag{14.23}$$

with

$$H = - S = k_B n V^{-1} \int dx_1 f_1 \ln f_1 + \tfrac{1}{2} k_B n^2 V^{-1} \int dx_1 \, dx_2 \, f_2 \ln \frac{f_2}{f_1 f_1} \; . \tag{14.24}$$

This is the Boltzmann H-function, taking completely into account the interactions, within the binary collision approximation. The correction term in (14.24) is the same as (14.8).

Had we used the functions F_1, F_2, G_2 instead of f_1, f_2, g_2 the H-function would have taken the form:

$$H = k_B\, n\, V^{-1} \int dx_1\, f_1 \;\ln f_1 + \tfrac{1}{2} k_B\, n^2\, V^{-1} \int dx_1\, dx_2 \left[F_2 \;\ln \frac{F_2}{F_1 F_1} - G_2 \right]$$

This form corresponds to the expression (14.15) for ΔS.

In sections 10 - 14 we studied the kinetic equations for spatially homogeneous gases. In the next paragraph we study the kinetic equations and the corresponding hydrodynamical equations for a spatially inhomogeneous nonideal gas.

15. THE BOLTZMANN EQUATION FOR A SPATIALLY INHOMOGENEOUS NONIDEAL GAS

In the framework of the binary collision approximation, we use the set of equations (2.11), (2.18) for the functions f_1, f_2. Assuming the complete weakening of the initial correlations, (2.18) can be transformed into

$$f_1(x_1, x_2, t) = f_1(X_1(-\tau), t-\tau)\, f_1(X_2(-\tau), t-\tau) + \int_0^\tau d\tau' \left[\left(\frac{\partial}{\partial t} + \boldsymbol{v}_1 \cdot \frac{\partial}{\partial \boldsymbol{r}_1} + \boldsymbol{v}_2 \cdot \frac{\partial}{\partial \boldsymbol{r}_2}\right) f_1(x_1, t)\, f_1(x_2, t)\right]_{X_{1,2}(-\tau'), t-\tau'} \tag{15.1}$$

where $X_{1,2}(-\tau)$ are the values of the coordinates and momenta of the particles at time $t-\tau$. We assumed $\boldsymbol{F} = 0$. For the monatomic gas this limitation is not essential for the derivation of the collision integral, as the work performed by the force $\boldsymbol{F}$ over a distance of order r_0 is negligible compared to $k_B T$.

Substituting (15.1) into (2.11) we obtain an expression of the collision integral, taking into account the spatial variation of the functions $f_1(x_1, t)$, $f_1(x_2, t)$ over the range of the interatomic forces as well as the retardation of these functions.

We shall consider several particular cases.

(1) The distribution function is spatially homogeneous, and the retardation is neglected. We then obtain the Boltzmann equation for the ideal gas, (section 10,11).

(2) The distribution function is spatially homogeneous, but the retardation is taken into account. We then obtain the Boltzmann equation for the spatially homogeneous nonideal gas (sections 12 - 14).

(3) The retardation is neglected, but the spatial inhomogeneity of f_1 over distances of order r_0 is retained. We then obtain a generalized Boltzmann

equation, due to Bogolyubov (Eq. (9.17) of ref. [4]). For the hard-sphere model, the latter equation reduces to the Enskog generalization of the Boltzmann equation (chapter 16 of ref. [2])[†].

The difference between these equations and the Boltzmann equation is manifest, for instance, in the hydrodynamic approximation. Using the generalized equations, we obtain in the hydrodynamic equations corrections to the equation of state of the ideal gas and to the transport coefficients in the form of density expansions.

It must be kept in mind that the Enskog and Bogolyubov generalizations were obtained within the binary collision approximation. As a result, the corrections of second and higher order in the density to the transport coefficients and to the equation of state are incorrect. Indeed, the neglected triple and higher order collisions give contributions of the same order.

In the kinetic equation for inhomogeneous gases, in the binary collision approximation, it is sufficient to take into account terms of first order in r_0/l, i.e. the terms of first order in the density.

The Enskog and Bogolyubov generalizations of the Boltzmann equation do not take into account the retardation effect. As a result, in going over to the hydrodynamical equations, they do not yield interaction corrections to the functions under the $\partial/\partial t$ sign. For instance, we do not obtain correction to the internal energy. In order to obtain all the first order corrections to the kinetic equation, we must take account of both the spatial inhomogeneity and the retardation, i.e., of all terms of order r_0/l and $\tau_0/\tau_{\mathbf{rel}}$.

(4) In addition to the results of sections 12 and 13, we need an evaluation of the terms of first order in the inhomogeneity. We proceed as follows.

We introduce into $f_2(\boldsymbol{r}_1, \boldsymbol{r}_2, \boldsymbol{p}_1, \boldsymbol{p}_2, t)$ the variables

$$\boldsymbol{r}_1 - \boldsymbol{r}_2 = \boldsymbol{r}_{12}, \quad (\boldsymbol{r}_1 + \boldsymbol{r}_2)/2 = \boldsymbol{r}_1 - (\boldsymbol{r}_{12}/2)$$

and expand the result in powers of $\boldsymbol{r}_{12}(\partial/\partial \boldsymbol{r}_1)$. To first order we have

$$\begin{aligned} f_2(x_1, x_2, t) &\equiv f_2(\boldsymbol{r}_{12}, \tfrac{1}{2}(\boldsymbol{r}_1 + \boldsymbol{r}_2), \boldsymbol{p}_1, \boldsymbol{p}_2, t) \\ &= \left(1 - \tfrac{1}{2}\boldsymbol{r}_{12} \cdot \frac{\partial}{\partial \boldsymbol{r}_1}\right) f_2(\boldsymbol{r}_{12}, \boldsymbol{r}_1, \boldsymbol{p}_1, \boldsymbol{p}_2, t) \qquad (15.2) \end{aligned}$$

We write an equation for the function

$$\begin{aligned} &f_2(\boldsymbol{r}_{12}, \boldsymbol{r}_1, \boldsymbol{p}_1, \boldsymbol{p}_2, t) \\ &\qquad \equiv g_2(\boldsymbol{r}_{12}, \boldsymbol{r}_1, \boldsymbol{p}_1, \boldsymbol{p}_2, t) + f_1(\boldsymbol{r}_1, \boldsymbol{p}_1, t) f_1(\boldsymbol{r}_1, \boldsymbol{p}_2, t) \end{aligned}$$

[†] **See also reference [68]. (Translator.)**

From (2.18) we obtain, to first order if the inhomogeneity:

$$\left[\frac{\partial}{\partial t} + \frac{1}{2}(\boldsymbol{v}_1 + \boldsymbol{v}_2)\cdot\frac{\partial}{\partial \boldsymbol{r}_1} + (\boldsymbol{v}_1 - \boldsymbol{v}_2)\cdot\frac{\partial}{\partial \boldsymbol{r}_{12}} - \frac{\partial \Phi_{12}}{\partial \boldsymbol{r}_1}\cdot\left(\frac{\partial}{\partial \boldsymbol{p}_1} - \frac{\partial}{\partial \boldsymbol{p}_2}\right)\right] f_2(\boldsymbol{r}_1, \boldsymbol{r}_{12}, \boldsymbol{p}_1, \boldsymbol{p}_2, t)$$
$$= \left[\frac{\partial}{\partial t} + \frac{1}{2}(\boldsymbol{v}_1 + \boldsymbol{v}_2)\cdot\frac{\partial}{\partial \boldsymbol{r}_1}\right] f_1(\boldsymbol{r}_1, \boldsymbol{p}_1, t)\, f_1(\boldsymbol{r}_1, \boldsymbol{p}_2, t) \quad . \tag{15.3}$$

Assuming complete weakening of the initial correlations, the solution of this equation is

$$f_2(\boldsymbol{r}_1, \boldsymbol{r}_{12}, \boldsymbol{p}_1, \boldsymbol{p}_2, t)$$
$$= f_1(\boldsymbol{r}_1 - \tfrac{1}{2}(\boldsymbol{v}_1 + \boldsymbol{v}_2)\tau, \boldsymbol{P}_1(-\tau), t-\tau)\, f_1(\boldsymbol{r}_1 - \tfrac{1}{2}(\boldsymbol{v}_1 + \boldsymbol{v}_2)\tau, \boldsymbol{P}_2(-\tau), t-\tau)$$
$$+ \int_0^\tau d\tau' \left[\frac{\partial}{\partial t} + \frac{1}{2}(\boldsymbol{v}_1 + \boldsymbol{v}_2)\cdot\frac{\partial}{\partial \boldsymbol{r}_1}\right] f_1(\boldsymbol{r}_1, \boldsymbol{P}_1(-\tau'), t)\, f_1(\boldsymbol{r}_1, \boldsymbol{P}_2(-\tau'), t) .$$

In the second term on the right-hand side, we only retained contributions of first order in the retardation and the inhomogeneity. In the first term, we expand in powers of τ and retain the contributions of zeroth and first order; in the second term we integrate by parts and go to the limit $\tau \to \infty$. We then find:

$$f_2(\boldsymbol{r}_1, \boldsymbol{r}_{12}, \boldsymbol{p}_1, \boldsymbol{p}_2, t) = f_1(\boldsymbol{r}_1, \boldsymbol{P}_1(-\infty), t)\, f_1(\boldsymbol{r}_1, \boldsymbol{P}_2(-\infty), t)$$
$$- \left(\frac{\partial}{\partial t} + \frac{1}{2}(\boldsymbol{v}_1 + \boldsymbol{v}_2)\cdot\frac{\partial}{\partial \boldsymbol{r}_1}\right)\int_0^\infty d\tau\, \tau \frac{\partial}{\partial \tau} f_1(\boldsymbol{r}_1, \boldsymbol{P}_1(-\tau), t)\, f_1(\boldsymbol{r}_1, \boldsymbol{P}_2(-\tau), t) \tag{15.4}$$

On substituting this expression into (15.2), we only retain terms of first order in r_0/l, $\tau_0/\tau_{\mathbf{rel}}$. Therefore, the operator $(\boldsymbol{r}_{12}/2)\cdot(\partial/\partial \boldsymbol{r}_1)$ only acts on the first term in (15.4). The resulting expression of $f_2(\boldsymbol{r}_1, \boldsymbol{r}_2, \boldsymbol{p}_1, \boldsymbol{p}_2, t)$ consists of a sum of three terms. The corresponding collision integral (2.12) can thus be represented in the form:

$$I(x_1, t) = I_{(1)} + I_{(2)} + I_{(3)} . \tag{15.5}$$

The first term is defined as follows:

$$I_{(1)}(x_1, t) = n\int dx_2 \frac{\partial \Phi_{12}}{\partial \boldsymbol{r}_1}\cdot\frac{\partial}{\partial \boldsymbol{p}_1} f_1(\boldsymbol{r}_1, \boldsymbol{P}_1(-\infty), t)\, f_1(\boldsymbol{r}_1, \boldsymbol{P}_2(-\infty), t) . \tag{15.6}$$

It differs from (10.5) in the dependence of f_1 on $\boldsymbol{r}_1$ The second term is:

$$I_{(2)}(x_1, t) = -n \int_0^\infty d\tau \int dx_2 \frac{\partial \Phi_{12}}{\partial \boldsymbol{r}_1} \cdot \frac{\partial}{\partial \boldsymbol{p}_1} \left(\frac{\partial}{\partial t} + \frac{1}{2} (\boldsymbol{v}_1 + \boldsymbol{v}_2) \cdot \frac{\partial}{\partial \boldsymbol{r}_1} \right) \cdot$$

$$\cdot \, \tau \frac{\partial}{\partial \tau} f_1(\boldsymbol{r}_1, \boldsymbol{P}_1(-\tau), t) \; f_1(\boldsymbol{r}_1, \boldsymbol{P}_2(-\tau), t) \,. \tag{15.7}$$

In the spatially homogeneous case, this term reduces to (12.6). Finally,

$$I_{(3)}(x_1, t) = -\frac{1}{2} n \int dx_2 \frac{\partial \Phi_{12}}{\partial \boldsymbol{r}_1} \cdot \frac{\partial}{\partial \boldsymbol{p}_1} \left(\boldsymbol{r}_{12} \cdot \frac{\partial}{\partial \boldsymbol{r}_1} \right)$$

$$\left[f_1(\boldsymbol{r}_1, \boldsymbol{P}_1(-\infty), t) f_1(\boldsymbol{r}_1, \boldsymbol{P}_2(-\infty), t) - f_1(\boldsymbol{r}_1, \boldsymbol{p}_1, t) \, f_1(\boldsymbol{r}_1, \boldsymbol{p}_2, t) \right] . \tag{15.8}$$

The presence of the second term in the square brackets is due to the fact that the collision integral is defined in terms of g_2, not f_2. This contribution is cancelled by a similar term in the expression of the average force on the left-hand side of the kinetic equation (see (2.10), (2.11). Thus, in Eq. (2.11) we replace $\boldsymbol{F}$ by $\boldsymbol{F}_0$.

We now use the kinetic equation with the collision term (15.5) - (15.8) for the derivation of the hydrodynamic equations of the nonideal gas. We consider the contribution of the collision integrals $I_{(1)}, I_{(2)}, I_{(3)}$ to the balance equations of the mass, momentum and energy densities. We thus consider the integrals $\int d\boldsymbol{p}\, \phi I$ for $\phi = 1, \boldsymbol{p}, p^2/2m$. The integral $I_{(1)}$ has the same property as (11.5), even for inhomogeneous systems:

$$n \int d\boldsymbol{p}_1 \; \phi(\boldsymbol{p}_1) \, I_{(1)}(x_1, t) = 0 \,, \quad \text{for} \quad \phi = 1, \boldsymbol{p}_1, p_1^2/2m \,. \tag{15.9}$$

Thus, the integral $I_{(1)}$ does not contribute to the transport equations. It does contribute, however, to the definition of the viscous pressure tensor π_{ij} and to the heat flow vector [5, 9].

Just as for the homogeneous case, the integral $I_{(2)}$ has the property

$$n \int d\boldsymbol{p}_1 \, \phi(\boldsymbol{p}_1) \, I_{(2)}(x_1, t) = 0 \,, \text{ for } \quad \phi = 1, \boldsymbol{p}_1 \,. \tag{15.10}$$

It does not contribute to the equations of mass and momentum transport.

In the number density equation, the integral $I_{(3)}$ gives no contribution either, because

$$n \int d\boldsymbol{p}_1 \, \phi(\boldsymbol{p}_1) \, I_{(3)}(x_1, t) = 0 \,, \quad \text{for} \quad \phi = 1 \,. \tag{15.11}$$

For $\phi = \boldsymbol{p}_1$, the integral $I_{(3)}$ gives a non-vanishing result, which contributes to the momentum transport equation. Multiplying (15.8) by $n\boldsymbol{p}_1$ and integrating by parts over $\boldsymbol{p}_1$ we obtain †

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† **We cancel the second term in the square brackets of (15.8) and replace F by F_0 (see Eq. (15.8)**

$$n \int d\boldsymbol{p}_1 \, p_{1i} \, I_{(3)}(x_1, t) = -\frac{\partial \Delta P_{ij}}{\partial r_{1j}} \ . \tag{15.12}$$

We introduced here the symbol ΔP_{ij} to denote the correction due to the interactions to the pressure tensor P_{ij}. It is defined as follows:

$$\Delta P_{ij} = -\tfrac{1}{2} n^2 \int d\boldsymbol{r}\, d\boldsymbol{p}_1 \, d\boldsymbol{p}_2 \, \frac{r_i r_j}{r} \frac{\partial \Phi}{\partial r} \, f_1(\boldsymbol{r}_1, \boldsymbol{P}_1(-\infty), t) \, f_1(\boldsymbol{r}_1, \boldsymbol{P}_2(-\infty), t) \tag{15.13}$$

where $\boldsymbol{r} = \boldsymbol{r}_1 - \boldsymbol{r}_2$. We used the fact that

$$\frac{\partial \Phi}{\partial r_i} = \frac{r_i}{r} \frac{\partial \Phi}{\partial r} \ .$$

Recalling that the viscous pressure tensor is defined by $\pi_{ij} = P_{ij} - \delta_{ij} p$, we represent the tensor ΔP_{ij} in the form:

$$\Delta P_{ij} = \delta_{ij} \Delta p + \Delta \pi_{ij} \ . \tag{15.14}$$

From (15.14) and (15.13) we obtain the contribution of the interactions to the scalar pressure:

$$\Delta p = -\tfrac{1}{6} n^2 \int d\boldsymbol{r}\, d\boldsymbol{p}_1 \, d\boldsymbol{p}_2 \, r\, \Phi'(r) \, f_1(\boldsymbol{r}_1, \boldsymbol{P}_1(-\infty), t) \, f_1(\boldsymbol{r}_1, \boldsymbol{P}_2(-\infty), t) . \tag{15.15}$$

In the present approximation for ΔP_{ij}, Δp, it is sufficient to calculate these quantities in the local equilibrium approximation. Indeed, the deviations from local equilibrium contribute terms of order $\varepsilon \varepsilon_h$, where ε_h is a small hydrodynamic parameter. In this approximation, we obtain from (15.13):

$$\Delta P_{ij} = \delta_{ij} \Delta_p \ , \qquad \Delta \pi_{ij} = 0 \ , \tag{15.16}$$

with

$$\Delta p = -\tfrac{1}{6} n^2 \int d\boldsymbol{r}\, d\boldsymbol{p}_1 \, d\boldsymbol{p}_2 \, r\, \Phi'(r) \, f_1^{(0)}(\boldsymbol{P}_1(-\infty)) \, f_1^{(0)}(\boldsymbol{P}_2(-\infty)) \ . \tag{15.17}$$

Substituting the Maxwell distribution and using (10.12), we may perform the momentum integration:

$$\begin{aligned} \Delta p &= -\tfrac{2}{3} \pi n^2 \int dr\, r^3 \, \Phi'(r) \, \exp(-\Phi/k_{\mathbf{B}}T) \\ &= \tfrac{2}{3} \pi n^2 k_{\mathbf{B}} T \int_0^\infty dr\, r^3 \, \frac{\partial}{\partial r} \Big[\exp(-\Phi/k_{\mathbf{B}}T) - 1 \Big] \end{aligned} \tag{15.18}$$

Integrating by parts over r, we finally obtain

$$\Delta p = -2\pi n^2 k_{\mathbf{B}} T \int_0^\infty dr\, r^2 \, \Big[\exp(-\Phi/k_{\mathbf{B}}T) - 1 \Big] \ . \tag{15.19}$$

This expression differs from (8.15), which was obtained in equilibrium, because in (15.19) the density and temperature are local quantities, depending on

the position and the time.

For the model of weakly attracting hard spheres, we obtain

$$\Delta p = n^2 k_B T \left(b - \frac{a}{k_B T} \right) , \tag{15.20}$$

where a and b are the van der Waals constants, defined locally.

We now consider the contribution of the collision integrals $I_{(2)}$, $I_{(3)}$ to the energy balance equation. We multiply (15.7) by $np_1^2/2m$ and integrate over $\boldsymbol{p}_1$. After performing transformations analogous to those of section 12 we obtain

$$n \int d\boldsymbol{p}_1 \frac{p_1^2}{2m} I_{(2)} = -\tfrac{1}{2} n^2 \int d\boldsymbol{r}\, d\boldsymbol{p}_1\, d\boldsymbol{p}_2 \Phi_{12} \left(\frac{\partial}{\partial t} + \tfrac{1}{2} (\boldsymbol{v}_1 + \boldsymbol{v}_2) \cdot \frac{\partial}{\partial \boldsymbol{r}_1} \right) \cdot$$

$$\cdot f_1(\boldsymbol{r}_1, \boldsymbol{P}_1(-\infty), t)\; f_1(\boldsymbol{r}_1, \boldsymbol{P}_2(-\infty), t) . \tag{15.21}$$

For spatially homogeneous systems, this expression reduces to (12.15).

In the local equilibrium approximation, (15.21) becomes

$$n \int d\boldsymbol{p}_1 \frac{p_1^2}{2m} I_{(2)} = -\left(\frac{\partial \Delta U}{\partial t} + \frac{\partial}{\partial \boldsymbol{r}_1} \cdot (\boldsymbol{u} \Delta U) \right) , \tag{15.22}$$

where

$$\Delta U = 2\pi n^2 \int_0^\infty dr\, r^2\, \Phi\, \exp\left(-\Phi / k_B T \right) , \tag{15.23}$$

is the contribution of the interactions to the internal energy, and $\boldsymbol{u}$ is the local average velocity.

For the interaction potential (8.18) this equation reduces to

$$\Delta U = -n^2 a \tag{15.24}$$

where a is the van der Waals constant.

We now consider the contribution of $I_{(3)}$ to the energy balance equation. We multiply (15.8) by $n p_1^2/2m$, integrate by parts over $\boldsymbol{p}_1$ and symmetrize:

$$n \int d\boldsymbol{p}_1 \frac{p_1^2}{2m} I_{(3)} = \tfrac{1}{4} n^2 \frac{\partial}{\partial r_j} \int d\boldsymbol{r}\, d\boldsymbol{p}_1\, d\boldsymbol{p}_2 (v_{1i} + v_{2i}) \frac{r_i r_j}{r} \Phi'(r) \cdot$$

$$\cdot f_1(\boldsymbol{r}_1, \boldsymbol{P}_1(-\infty), t)\; f_1(\boldsymbol{r}_1, \boldsymbol{P}_2(-\infty), t) . \tag{15.25}$$

In the local equilibrium approximation we find

$$n \int d\boldsymbol{p}_1 \frac{p_1^2}{2m} I_{(3)} = -\frac{\partial}{\partial \boldsymbol{r}} \cdot (\boldsymbol{u} \Delta p) , \tag{15.26}$$

where $\boldsymbol{u}$ is the average velocity and Δp is defined by (15.19).

After these calculations, we can write the hydrodynamical equations for the nonideal gas, in the binary collision approximation, in their final form. From (7.2), (7.9), (7.14), (15.12), (15.22), (15.26) we find:

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial \boldsymbol{r}} \cdot \rho \boldsymbol{u} = 0 \,, \tag{15.27}$$

$$\frac{\partial}{\partial t} \rho u_i + \frac{\partial}{\partial r_j} [\rho\, u_i u_j + \delta_{ij}\, p + \pi_{ij}] = \frac{\rho}{m} F_{0i} \,, \tag{15.28}$$

$$\frac{\partial}{\partial t} (\tfrac{1}{2} \rho u^2 + U) + \frac{\partial}{\partial r_j} [u_j (\tfrac{1}{2} \rho u^2 + U + p) + \pi_{ij} u_i + S_j] = \frac{\rho}{m} \boldsymbol{F}_0 \cdot \boldsymbol{u} \,. \tag{15.29}$$

where

$$p = p_{id} + \Delta p \,, \quad U = U_{id} + \Delta U \tag{15.30}$$

are the pressure and the internal energy of the nonideal gas, defined, in the binary collision approximation by (15.18) (or (15.19)) and (15.23).

The viscous pressure tensor π_{ij} and the heat flow vector $\boldsymbol{S}$ are defined, in the present approximation, by (7.17) and (7.18).

Clearly, instead of the equations for ρ, $\rho \boldsymbol{u}$, $(\rho u^2/2) + U$, we may just as well write equations for the functions ρ, $\boldsymbol{u}$, T, because the pressure p and the internal energy can be expressed in terms of ρ and T.

These hydrodynamical equations were obtained from the kinetic equation

$$\left(\frac{\partial}{\partial t} + \boldsymbol{v}_1 \cdot \frac{\partial}{\partial \boldsymbol{r}_1} + \boldsymbol{F}_0 \cdot \frac{\partial}{\partial \boldsymbol{p}_1}\right) f_1 = I_{(1)} + I_{(2)} + I_{(3)} \,, \tag{15.31}$$

in which the collision integral takes account of the retardation effect and of the spatial inhomogeneity. If these effects are neglected, i.e. if $I_{(2)} = I_{(3)} = 0$, then in the equations of hydrodynamics $\Delta p = 0$, $\Delta U = 0$, and thus $p = p_{\text{id}}$, $U = U_{\text{id}}$. Hence, on the basis of the ordinary Boltzmann equation, one obtains the hydrodynamical equations of the ideal gas.

In the derivation of these hydrodynamical equations, we retained only terms which are linear in both the density parameter $\varepsilon = n r_0^3$ and the hydrodynamic parameter $\varepsilon_h = l/L$ (where L is a characteristic length of the system). The problem of higher approximations will be studied in the next chapter. Here we only note that the expansion of the kinetic equation in powers of the hydrodynamic parameter is effective only when the velocity distribution is close to the local Maxwell distribution. For large deviations from this state, the velocity distribution may be very badly distorted [65]. In such cases, the hydrodynamic description of the processes may be less effective.

CHAPTER 3

Kinetic Equations for Dense Gases

16 PROBLEMS OF THE KINETIC THEORY OF DENSE GASES

The collision integral in the Boltzmann kinetic equation for the non-ideal monatomic gas (10.5) is defined by the expression (10.4) of the two-particle distribution function. Let us recall the assumptions under which this expression was obtained.

(1) Neglect of the collisions involving three or more particles. By this assumption the Bogolyubov hierarchy reduces to a closed set of equations (2.11), (2.18) for the distribution functions f_1, f_2.

(2) In solving Eq. (2.18) for the distribution function f_2, use is made of the complete weakening of the initial correlations: the Bogolyubov condition.

(3) In solving the equation for f_2 the time delay and the spatial inhomogeneity of the one-particle distribution function f_1 are disregarded. Giving up this assumption (sections 12-15) leads, in the framework of the pair-collision approximation, to the kinetic and hydrodynamical equations of the non-ideal gas.

(4) It was assumed (though not explicitly) that the collision process of each particle with its neighbours is continuous within a physically infinitesimal volume. This assumption will be dropped in chapter 4.

The presence of a small parameter ε — the density parameter — would suggest the construction of the kinetic equations, with account of triple and higher order collisions, by expanding the distribution f_2 as a power series in ε.

In his work [4], Bogolyubov developed a method, using the condition of complete weakening of initial many-particle correlations, by which the form of the distribution function f_2 can be obtained in each order of approximation in the parameter ε.

The expression (10.4) is the zeroth approximation in ε. The next

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approximation contains terms of two kinds. One group describes the influence of the three-body collisions on the dissipative characteristics of the gas. The second group describes the influence of pair collisions on the non-dissipative properties of the gas, e.g., the corrections to the thermodynamic functions due to the pair interactions of the particles. This is equivalent to retaining in the thermodynamic functions the terms of first order in $\tau_0/\tau_{\mathbf{rel}}$, r_0/l.

Using the expression for the function f_2 containing the terms linear in the density, it is possible to obtain refined expressions for the collision integral in the kinetic equation. This was done by Choh and Uhlenbeck [5].

If the next approximation in the density is retained for f_2, a more general kinetic equation can be obtained. It accounts for the effect of four-body collisions on the dissipative properties and of three-body collisions on the non-dissipative quantities.

Thus it seems that a general method is available for the construction of kinetic equations for gases to an arbitrary order in the density. In each order one uses only the condition of the weakening of initial correlations in times much shorter than $\tau_{\mathbf{rel}}$, and the continuity of the collision. However, the realization of this programme is faced with serious difficulties of principle.

The independent investigations of a number of authors: Weinstock [19], Goldman and Frieman [20], and Dorfman and Cohen [21] (see also Cohen [22]) showed that the contributions to the collision integral to higher orders in the density contain divergent time-integrals. It is interesting to note that the character of the divergence is different in two-dimensional and in three-dimensional gas models.

In two-dimensional models already the first density correction (the contribution of ternary collisions) leads to a logarithmic divergence ($\ln t/\tau_0$) in the collision integral. The terms of order ε^n $(n \geqslant 2)$ lead to divergences of the type $(t/\tau_0)^{n-1}$.

In three dimensions the first approximation in ε (three-body collisions) is convergent. The terms of order $\varepsilon^n (n \geqslant 3)$ in f_2 lead to divergences of the form $(t/\tau_0)^{n-2}$.

This shows that the construction of kinetic equations for dense gases by a direct density expansion is impossible. There exist a few papers [22] in which it was shown, in special cases, that the logarithmic divergence can be suppressed by summing the most divergent diagrams in each order in the density. This leads to a cut-off of the time integrals at $\tau_{\mathbf{rel}}$. As a result the collision integral becomes proportional to $\ln(\tau_{\mathbf{rel}}/\tau_0) \sim \ln(1/\varepsilon)$ which depends non-analytically on the density. As a result, a virial expansion of the kinetic coefficients is impossible.

The problem of obtaining kinetic equations for dense gases can be solved in an alternative way, without summation of divergent density diagrams. One must then

give up the condition of complete weakening of inital correlations in times much smaller than τ_{rel} (section 18).

It was pointed out several years ago that the condition of complete weakening of the correlations is only an approximation (see, e.g., the papers by Sandri, and by Hopfield and Bastin [64]). Only the influence of short scale fluctuations is weakened, more precisely of those fluctuations for which, in the binary collision approximation, the correlation length and the correlation time are such that

$$r_{cor} \ll l \ , \ \tau_{cor} \ll \tau_{rel} \ . \tag{16.1}$$

In the general case, there is only a partial weakening of the correlations. The long-timescale fluctuations do not decay quickly enough and must therefore be considered in the construction of the kinetic theory. Their role leads, in particular, to the fact that the one-particle distribution function f_1, obeying the kinetic equation, is not truly deterministic (section 19).

The condition separating the region of fast and of slow fluctuations can be introduced as follows [23,24].

Consider the approximation of pair collisions. We denote by l_ϕ^3, τ_ϕ the volume and the time interval which can be regarded as physically infinitesimal in the Boltzmann equation. Hence, $\tau_\phi = l_\phi / v_T$.

The quantity l_ϕ is defined from the condition of continuity of the collision process: the interval between successive collisions involving an arbitrary particle within the physically infinitesimal volume equals τ_ϕ, i.e.

$$\frac{\tau_{rel}}{n\, l_\phi^3} = \tau_\phi = \frac{l_\phi}{v_T} \ . \tag{16.2}$$

Hence, using the results of section 2,

$$l_\phi = \varepsilon^{\frac{1}{2}}\, l, \quad \tau_\phi = \varepsilon^{\frac{1}{2}}\, \tau_{rel}, \quad l_\phi^3 n \sim \varepsilon^{-\frac{1}{2}} \gg 1 \ . \tag{16.3}$$

We shall call short-scale fluctuations those fluctuations for which $\tau_{cor} \leqslant \tau_\phi$, $r_{cor} \leqslant l_\phi$. From (16.3) it follows that they decay in a time of order $\varepsilon^{\frac{1}{2}}\tau_{rel} \ll \tau_{rel}$.

We use similar arguments for three- and four-body collisions. We denote the corresponding relaxation times by $\tau_{rel}^{(3)}, \tau_{rel}^{(4)}$. They are related to τ_{rel} as follows:

$$\tau_{rel}^{(3)} = \tau_{rel}\, \varepsilon \ , \quad \tau_{rel}^{(4)} = \tau_{rel}\, \varepsilon^2 \ .$$

Using instead of τ_{rel} in Eq. (16.2) the quantities $\tau_{rel}^{(3)}$, $\tau_{rel}^{(4)}$ we obtain

$$\tau_\phi^{(3)} \sim \varepsilon^{\frac{1}{4}}\, \tau_{rel} \ll \tau_{rel} \ , \quad l_\phi^{(3)} \sim \varepsilon^{\frac{1}{4}}\, l \ll l \ .$$

But $\tau_\phi^{(4)} \sim \tau_{rel}$, $l_\phi^{(4)} \sim l$; hence when considering four-body collisions, there exist correlations for which $\tau_{cor} \sim \tau_{rel}$, $l_{cor} \sim l$. Hence the collision integral

will depend on $\tau_{\mathbf{rel}}$ (or l) : this introduces an additional dependence on the density.

Note that for the two-dimensional case, already for triple collisions (in 16.2) $\tau_{\mathbf{rel}}/nl_{\phi}^{3} \simeq \tau_{\phi} = l_{\phi}/v_{T}\tau_{\phi} \sim \tau_{\mathbf{rel}}$, $l_{\phi} \sim l$. Hence the triple-collision integral cannot be constructed without accounting for the correlations with $\tau_{\mathbf{cor}} \sim \tau_{\mathbf{rel}}$, $l_{\mathbf{cor}} \sim l$. This explains why the density expansion breaks down earlier in the two-dimensional case.

Thus, the definition of the limits separating fast from slow fluctuations depends on the approximation considered.

The previous discussion shows that, in a time $\tau \ll \tau_{\mathbf{rel}}$, there is only a partial weakening of the correlations which enter the kinetic equation for dense gases.

By using the condition of partial weakening of the pair correlations, the Liouville equation does not lead to the Boltzmann equation. The simplification of the original Liouville equation consists only in the fact that, instead of an equation for the exact distribution f_N, we get an approximate equation for the distribution $\tilde{f}_N$ smoothed over an interval τ_{ϕ}, l_{ϕ}, (section 18).

From the equation for $\tilde{f}_N$ there follows again a hierarchy of equations, but now for the smoothed distribution functions $\tilde{f}_1 = f_1$, $\tilde{f}_2$, $\tilde{f}_3$,... . This hierarchy differs from the BBGKY hierarchy; it is already dissipative, through the pair correlations, and all functions are slowly varying.

In section 18 it is shown how this hierarchy can be used for the construction of kinetic equations for dense gases.

In studying the kinetic equations for gases in various approximations it is assumed that long-scale fluctuations ($\tau_{\mathbf{cor}} \geqslant \tau_{\mathbf{rel}}$, $l_{\mathbf{cor}} \geqslant l$) do not play any role. But these fluctuations do exist. They lead to the fact that the distribution functions for which the kinetic equations are derived are not strongly determined (section 19).

The study of the long-scale fluctuations aroused considerable interest in recent years in connection with the study of noise in various systems. This study can be done in two ways.

In the first method one starts from kinetic equations which are considered as Langevin equations with a random source. Such a method was used for gases by Kadomtsev [25], Gor'kov, Dzyaloshinsky and Pitaevsky [38] and by Kogan and Shulman [39].

The second method is based on the approximate solution of the hierarchy of equations for the distribution functions $f_1, f_2,...$ or of the corresponding equations for the Green functions [40-45] . In section 19 the long-scale fluctuations in gases will be calculated from the hierarchy of equations for the smoothed

distribution functions $\tilde{f}_1, \tilde{f}_2, \ldots$ in which the dissipation is introduced from the beginning through the pair collisions [23,24].

In recent times another method for the construction of generalized kinetic equations has been widely used. It is based, on the one hand, on the work of Prigogine, Balescu, Résibois, Hénin, George, Wallenborn, [46-48] and, on the other hand, on the work of Zubarev, Peletminsky, Tsukanov, Kalashnikov and Novikov [50-52]. In these papers the authors endeavour to derive directly the general kinetic equations for particles with arbitrary interactions.†)

In section 20 such a generalized kinetic equation will be derived, taking account both of dissipative and of non-dissipative terms. For this purpose the method applied in section 12 for the derivation of the non-ideal Boltzmann equation will be used. The generalized kinetic equation is much simpler than the Liouville equation, as it involves a one-particle distribution function. However, it is still quite complicated and its practical use requires further simplification. Nevertheless this method may be quite successful.

17. KINETIC EQUATIONS FOR NONIDEAL GASES TAKING INTO ACCOUNT TRIPLE COLLISIONS

In the previous section we mentioned that in Bogolyubov's work [4] a method was developed for the density expansion of the two-particle function f_2 for gases. In particular, in the case of spatially homogeneous gases the first two terms of the Bogolyubov density expansion of f_2 can be written as follows [4,5]:

$$\begin{aligned} f_2(x_1,x_2,t) &= S^{(2)}_{-\infty}(x_1,x_2)\, f_1 f_1 \\ &\quad + n\int_0^{\infty} d\tau \int dx_3\, S^{(2)}_{-\tau}(x_1,x_2)\Big\{(\hat{\theta}_{13}+\hat{\theta}_{23})\, S^{(3)}_{-\infty}(x_1,x_2,x_3) \\ &\quad - S^{(2)}_{-\infty}(x_1,x_2)\Big[\hat{\theta}_{13} S^{(2)}_{-\infty}(x_1,x_3) + \hat{\theta}_{23} S^{(2)}_{-\infty}(x_2,x_3)\Big]\Big\} f_1 f_1 f_1 \end{aligned} \tag{17.6}$$

Here $S^{(n)}_{-\infty}(x_1,\ldots,x_n)$ is the propagator for the n-particle system $x_1, x_2, \ldots, x_n$. The operator $\hat{\theta}_{ij}$ was defined in Eq. (1.6). The normalization of the functions f_1, f_2 was defined in Eq. (1.14).

For obtaining the collision integral, Eq. (17.1) must be substituted in (10.1). The first term in the expansion (17.1) yields a contribution describing the dissipative characteristics of the gas in the binary collision approximation and leads to the Boltzmann collision integral (10.5).

Among the two terms proportional to n, the first one (involving $S^{(3)}_{-\infty}$) describes the dissipative characteristics of the gas in the triple-collision

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† **A thorough discussion of the matters in this chapter is found in Résibois' and de Leener's book [68]. See also [67]. (Transl.)**

approximation. The second one takes account of the binary-collision corrections to the non-dissipative properties of the gas (e.g., to the internal energy).

Both terms were obtained in the kinetic equation due to Choh and Uhlenbeck [5]. We see that the Choh-Uhlenbeck equation does not describe the particle interactions quite consistently: in the dissipative terms the influence of both binary and triple collisions is included, whereas in the non-dissipative ones only binary collisions are retained.

As a result, in deriving the hydrodynamic equations with account of the triple collisions, Choh and Uhlenbeck are led to introduce additional terms in the expression of the pressure (ref. [5], p.264). These terms, which ensure the consistency of the approximation for the dissipative and the non-dissipative quantities, are not contained in the kinetic equation.

Let us consider the kinetic equation for a gas in the triple-collision approximation. We include the contribution of the particle interactions in both the dissipative and the non-dissipative quantities [24,53].

We rewrite the first three equations of the BBGKY hierarchy in the case of a spatially homogeneous gas in the absence of external forces $(\boldsymbol{F}_0 = 0)$

$$\frac{\partial f_1}{\partial t} = n\int dx_2\, \hat{\theta}_{12} f_2 \equiv I\,, \tag{17.2}$$

$$\frac{\partial f_2}{\partial t} + \hat{H}_2 f_2 = n\int dx_3 (\hat{\theta}_{13} + \hat{\theta}_{23})\, f_3 \equiv I_2\,, \tag{17.3}$$

$$\frac{\partial f_3}{\partial t} + \hat{H}_3 f_3 = n\int dx_4 (\hat{\theta}_{14} + \hat{\theta}_{24} + \hat{\theta}_{34})\, f_4 \equiv I_3\,. \tag{17.4}$$

In section 2 it was shown [see Eq. (2.20)] that in the binary collision approximation the right-hand side of Eq. (17.3) is

$$I_2 = \frac{\partial}{\partial t} f_1 f_1\,. \tag{17.4$_1$}$$

With the use of $(17.4)_1$ the equations (17.2), (17.3), in the pair-collision approximation, become a closed set of equations for f_1, f_2. Using, moreover, the Bogolyubov condition of weakening of the initial correlations to the first approximation in $\tau_0/\tau_{\mathbf{rel}}$ $(\tau_0 = r_0/v_T)$, one obtains Eq. (12.4). We re-write it here in the form:

$$\begin{aligned} f_2^{\mathrm{bc}} &= S_{-\infty}^{(2)}\; f_1(\boldsymbol{p}_1,t)\, f_1(\boldsymbol{p}_2,t) \\ &- \frac{\partial}{\partial t}\int_0^\infty d\tau\, \frac{d}{d\tau}\, S_{-\tau}^{(2)}\; f_1(\boldsymbol{p}_1,t)\, f_1(\boldsymbol{p}_2,t) \equiv f_2^0 + f_2^1\,. \end{aligned} \tag{17.5}$$

Boltzmann collision integral. The second term accounts for the contribution of the binary collisions to the non-dissipative characteristics.

Consider now the triple-collision approximation. In this case we express the function f_4 on the right-hand side of Eq. (17.4) through the functions f_1, g_2, g_3, g_4. The term involving g_4 is neglected, as well as other terms which differ from zero only when four particles are simultaneously together. Thus the function f_4, acted upon in Eq. (17.4) by the operator $\hat{\theta}_{14}$, is approximated by

$$\begin{aligned} f_4^{(1,4)} &= f_1(1)f_1(2)f_1(3)f_1(4) + f_1(1)f_1(2)g_2(3,4) + \\ &+ f_1(1)f_1(3)g_2(2,4) + \left[f_1(2)f_1(3) + g_2(2,3)\right]g_2(1,4) + \\ &+ f_1(2)g_3(1,3,4) + f_1(3)g_3(1,2,4). \end{aligned}$$

Here we retain only terms which are non-zero when the two particles 1 and 4 are close together. As will be seen, they compensate the $\tau \to \infty$ divergence in the solution of the homogeneous equation (17.4) in the terms of order $\tau(\partial/\partial t)$. Retaining only these terms we obtain

$$f_4^{(1,4)} = f_2(1,4)\, f_1(2)\, f_1(3) \ .$$

Taking account of the analogous contributions to the other terms of the right-hand side of (17.4) and using Eq. (17.2), we obtain

$$I_3 = \frac{\partial}{\partial t} f_1 f_1 f_1 \tag{17.6}$$

and hence a closed set of equations is obtained for f_1, f_2, f_3.

From Eqs (17.4) and (17.6) together with the condition of complete weakening of the initial correlations, we obtain to first order in $\tau_0/\tau_{\mathbf{rel}}$:

$$\begin{aligned} f_3^{\mathrm{tc}}(x_1,x_2,x_3,t) &= S_{-\infty}^{(3)}\, f_1(\boldsymbol{p}_1,t) f_1(\boldsymbol{p}_2,t)\, f_1(\boldsymbol{p}_3,t) \\ &- \frac{\partial}{\partial t}\int_0^{\infty} d\tau\, \tau \frac{d}{d\tau} S_{-\tau}^{(3)}\, f_1(\boldsymbol{p}_1,t)\, f_1(\boldsymbol{p}_2,t)\, f_1(\boldsymbol{p}_3,t) \\ &\equiv f_3^0 + f_3^1 \ . \end{aligned} \tag{17.7}$$

This solution for f_3 is analogous to the solution (17.5) for the function f_2 in the binary collision approximation.

Consider now Eq. (17.3). The condition of the complete weakening of the initial weakening of the initial correlations of particles 1, 2, leads to the following expression to the second order in the density parameter $(nr_0^3)^2 \sim n r_0^3\, \tau_0/\tau_{\mathbf{rel}}$:

$$f_2(x_1,x_2,t;\tau) = S_{-\tau}^{(2)}(1,2)\left(1-\tau\frac{\partial}{\partial t}+\frac{\tau^2}{2}\frac{\partial^2}{\partial t^2}\right)f_1(\boldsymbol{p}_1,t)\ f_1(\boldsymbol{p}_2,t)$$
$$+\,n\int_0^\tau d\tau'\int dx_3\ S_{-\tau'}^{(2)}(1,2)(\hat\theta_{13}+\hat\theta_{23})\left(1-\tau'\ \frac{\partial}{\partial t}\right)f_3^{\mathbf{tr}}(x_1,x_2,x_3,t). \tag{17.8}$$

Using Eq. (17.2) for $f_1(\boldsymbol{p}_1,t)$, $f_1(\boldsymbol{p}_2,t)$ and taking the limit $\tau\to\infty$, we rewrite this expression as

$$f_2(x_1,x_2,t) = S_{-\infty}^{(2)}\,f_1\,f_1 + n\int_0^\infty d\tau\int dx_3\left(1-\tau\frac{\partial}{\partial t}\right)\left\{S_{-\tau}^{(2)}(1,2)(\hat\theta_{13}+\hat\theta_{23})\ f_3^{\mathbf{tr}}\right.$$
$$\left.-\ S_{-\infty}^{(2)}(1,2)\left[\hat\theta_{13}\,f_2^{\mathbf{bc}}(1,3)\,f_1(2)+\hat\theta_{23}\,f_2^{\mathbf{bc}}(2,3)\,f_1(1)\right]\right\}. \tag{17.9}$$

If we neglect the terms in $n\tau(\partial/\partial t)$, which describe the retardation effects in the triple-collision approximation, taking account of the fact that $S_{-\infty}^{(2)}=S_{-\tau-\infty}^{(2)}$, and substituting (17.5) and (17.7) (without retardation effects) into (17.9), the latter reduces to (17.1). Substituting this expression into the right-hand side of Eq. (17.2) we obtain the Choh-Uhlenbeck kinetic equation.

We come back to Eq. (17.9) and rewrite it in a form in which the contribution of the binary collisions, with account of the retardation, can be clearly recognized. We add and subtract in (17.9) the term

$$\int_0^\infty d\tau'\ S_{-\tau'}\ \frac{\partial}{\partial t}\ f_1\,f_1$$

and use Eq. (17.2) as well as the identity:

$$-\int_0^\infty d\tau\left(S_{-\tau}^{(2)}-S_{-\infty}^{(2)}\right)\frac{\partial}{\partial t}\,f_1\,f_1=\frac{\partial}{\partial t}\int_0^\infty d\tau\ \tau\ \frac{d}{dt}\,S_{-\tau}^{(2)}\ f_1\,f_1. \tag{17.10}$$

Finally, in the term containing $n\tau(\partial/\partial t)$ we change $S_{-\infty}^{(2)}(1,2)$ into $S_{-\tau-\infty}^{(2)}(1,2)$. We thus obtain the following expression for f_2 in the triple-collision approximation:

$$f_2^{\mathbf{tr}}(x_1,x_2,t) = f_2^{\mathbf{bc}}(x_1,x_2,t)$$
$$+\,n\int_0^\infty d\tau\int dx_3\,S_{-\tau}^{(2)}(1,2)\left[(\hat\theta_{13}+\hat\theta_{23})\,f_3^{\mathbf{tr}}(1,2,3)\right.$$
$$\left.-\ \hat\theta_{13}\ f_2^{\mathbf{bc}}(1,3)\,f_1(2)-\hat\theta_{23}\,f_2^{\mathbf{bc}}(2,3)\,f_1(1)\right]$$
$$-\,n\,\frac{\partial}{\partial t}\int_0^\infty d\tau\int dx_3\,\tau\,S_{-\tau}^{(2)}(1,2)\left\{(\hat\theta_{13}+\hat\theta_{23})\,f_3^0\right.$$
$$\left.-\ S_{-\infty}^{(2)}(1,2)\left[\hat\theta_{13}\,f_2^0(1,3)\,f_1(2)+\hat\theta_{23}\,f_2^0(2,3)\ f_1(1)\right]\right\}. \tag{17.11}$$

The functions f_2^0, f_3^0 are defined by Eqs (17.5), (17.7) without the time retardation:

$$f_2^0 = S_{-\infty}^{(2)} f_1(\boldsymbol{p}_1, t) f_1(\boldsymbol{p}_2, t) ,$$

$$f_3^0 = S_{-\infty}^{(3)} f_1(\boldsymbol{p}_1, t) f_1(\boldsymbol{p}_2, t) f_1(\boldsymbol{p}_3, t) . \tag{17.12}$$

The substitution of any one of the expressions (17.8), (17.9), (17.11) into the right-hand side of Eq. (17.21), yields expressions for the collision integral describing triple collisions and the time-retardation.

Let us show that from the approximate kinetic equation one can prove the conservation of the total (kinetic + potential) energy. The internal energy density thus obtained corresponds to the first three terms of its virial expansion.

We substitute f_2 from (17.11) into the expression

$$I^{\mathbf{tr}} = n \int dx_2 \, \theta_{12} f_2^{\mathbf{tr}} , \tag{17.13}$$

which defines the collision integral in the present approximation. From Eqs. (17.13), (17.11), (17.12) follows

$$n \int d\boldsymbol{p}_1 \; \phi(\boldsymbol{p}_1) I^{\mathbf{tr}}(\boldsymbol{p}_1, t) = 0 \quad \text{for} \quad \phi = 1, \boldsymbol{p}_1 . \tag{17.14}$$

These equations express the conservation of the particle number and momentum by the collisions.

We now multiply Eq. (17.13) by $n\boldsymbol{p}_1^2/2m$ and integrate over $\boldsymbol{p}_1$. After integration by parts and symmetrization we obtain

$$n \int d\boldsymbol{p}_1 (p_1^2/2m) I^{\mathbf{tr}} = -\frac{n^2}{2} \int d\boldsymbol{p}_1 \, d\boldsymbol{p}_2 \, d\boldsymbol{r}_{12} (\boldsymbol{v}_1 - \boldsymbol{v}_2) \cdot \frac{\partial \Phi_{12}}{\partial \boldsymbol{r}_{12}} f_2^{\mathbf{tr}} \tag{17.15}$$

where $\boldsymbol{r}_{12} = \boldsymbol{r}_1 - \boldsymbol{r}_2$. Consider the contribution of the various terms of (17.11) to Eq. (17.15). The first term defines the contribution of the binary collisions; hence, from the results (11.22) and (12.15), we can write

$$\left[n \int d\boldsymbol{p}_1 (p_1^2/2m) \, I^{\mathbf{tr}} \right]_1$$

$$= -\frac{\partial}{\partial t} \frac{n^2}{2} \int d\boldsymbol{r}_{12} \; d\boldsymbol{p}_1 \, d\boldsymbol{p}_2 \; \Phi_{12} \; f_1(\boldsymbol{P}_1(-\infty), t) \, f_1(\boldsymbol{P}_2(-\infty), t) . \tag{17.16}$$

Consider the contribution of the second term of (17.11). We introduce the notation

$$A(x_1, x_2, t) = \int dx_3 \left[(\hat{\theta}_{13} + \hat{\theta}_{23}) f_3^{\mathbf{tr}}(1, 2, 3) \right.$$

$$\left. - \hat{\theta}_{13} f_2^{\mathrm{bc}}(1, 3) f_1(2) - \hat{\theta}_{23} f_2^{\mathrm{bc}}(2, 3) f_1(1) \right] . \tag{17.17}$$

Thus from (17.15), (17.11) we find

$$\left[n \int d\boldsymbol{p}_1 \; (p_1^2/2m) \, I^{\mathbf{tr}} \right]_2$$

$$= -\frac{n^2}{2V} \int_0^\infty d\tau \int dx_1 \, dx_2 (\boldsymbol{v}_1 - \boldsymbol{v}_2) \cdot \frac{\partial \Phi_{12}}{\partial \boldsymbol{r}_{12}} S_{-\tau}^{(2)} \; A(x_1, x_2, t) . \tag{17.18}$$

Performing transformations analogous to those leading to Eq. (11.22), we obtain

$$\left[n\int d\boldsymbol{p}_1\,(p_1^2/2m)\,I^{\mathrm{tr}}\right]_2 = 0\,. \tag{17.19}$$

Consider finally the contribution of the third term of (17.11) to Eq. (17.15). We introduce the notation

$$B(x_1,x_2,t) = \int dx_3\Big\{(\hat\theta_{13}+\hat\theta_{23})f_3^{\mathrm{tr}}(1,2,3) - S_{-\infty}^{(2)}(1,2)\Big[\hat\theta_{13}f_2^{\mathrm{bc}}(1,3)f_1(2)+\hat\theta_{23}f_2^{\mathrm{bc}}(2,3)f_1(1)\Big]\Big\}\,. \tag{17.20}$$

From (17.15), (17.11) (17.20) we obtain :

$$\left[n\int d\boldsymbol{p}_1(p_1^2/2m)\,I^{\mathrm{tr}}\right]_3 = \frac{\partial}{\partial t}\frac{n^2}{2V}\int d\tau\int dx_1\,dx_2\,(\boldsymbol{v}_1-\boldsymbol{v}_2)\cdot\frac{\partial\Phi_{12}}{\partial\boldsymbol{r}_{12}}\,\tau S_{-\tau}^{(2)}(1,2)\,B(x_1,x_2,t)\,. \tag{17.21}$$

Performing transformations analogous to those leading to (12.15) we obtain

$$\left[n\int d\boldsymbol{p}_1(p_1^2/2m)\,I^{\mathrm{tr}}\right]_3 = -\frac{n^3}{2V}\frac{\partial}{\partial t}\int_0^\infty d\tau\int dx_1\,dx_2\,\Phi_{12}\,S_{-\tau}^{(2)}\,B(x_1,x_2,t)\,. \tag{17.22}$$

Adding together Eqs. (17.16), (17.19) and (17.22), and using (17.20) we obtain the final result :

$$\begin{aligned} n\int d\boldsymbol{p}_1\,(p_1^2/2m)\,I^{\mathrm{tr}} = -\frac{\partial}{\partial t}\frac{n^2}{2V}\int dx_1\,dx_2 &\cdot\Phi_{12}\Big\{S_{-\infty}^{(2)}f_1(1)f_1(2)+n\int_0^\infty d\tau\int dx_3\,S_{-\tau}^{(2)}(1,2)\\ &\cdot\Big[(\hat\theta_{13}+\hat\theta_{23})\,S_{-\infty}^{(3)}(1,2,3)\\ &-S_{-\infty}^{(2)}(1,2)\Big(\hat\theta_{13}S_{-\infty}^{(2)}(1,3)+\hat\theta_{23}S_{-\infty}^{(2)}(2,3)\Big)\Big]\\ &\cdot f_1(1)f_1(2)f_1(3)\Big\}\,. \end{aligned} \tag{17.23}$$

The expression enclosed in curly brackets in Eq. (17.23) is identical to Eq. (17.1) which defines the two-particle distribution function f_2 to second order in the density. Hence, from the kinetic equation (17.2) together with the collision integral I^{tr} we obtained the conservation of energy :

$$\frac{\partial}{\partial t}\left\{n\int d\boldsymbol{p}_1\,(p_1^2/2m)\,f_1+\frac{n^2}{V}\int dx_1\,dx_2\,\Phi_{12}\,f_2\right\}=0\,, \tag{17.24}$$

where the function f_2 is defined by Eq. (17.1). From (17.14) and (17.1) follows the expression of the internal energy density in the triple collision approximation:

$$U = n\int d\boldsymbol{p}_1\,\frac{p_1^2}{2m}\,f_1 + \frac{n^2}{2V}\int dx_1\,dx_2\,\Phi_{12}S^{(2)}_{-\infty}\,f_1f_1$$

$$+\frac{n^3}{2V}\int dx_1\,dx_2\,\Phi_{12}\left\{\int_0^{\infty}d\tau\int dx_3\,S^{(2)}_{-\tau}(1,2)\Big[(\hat{\theta}_{13}+\hat{\theta}_{23})\,S^{(3)}_{-\infty}\right.$$

$$\left.-S^{(2)}_{-\infty}(1,2)\,\left(\hat{\theta}_{13}S^{(2)}_{-\infty}(1,3)+\hat{\theta}_{23}S^{(2)}_{-\infty}(2,3)\right)\Big]\,f_1\,f_1\,f_1\right\}. \tag{17.25}$$

This expression defines the first three terms of the virial expansion of the internal energy for a nonequilibrium state of the gas.

In the local equilibrium approximation (17.25) goes over into

$$U = n\frac{3}{2}k_BT+\frac{n^2}{2V}\int d\boldsymbol{r}_1\,d\boldsymbol{r}_2\,\Phi_{12}\,e^{-\Phi_{12}/k_BT}+\frac{n^3}{2V}\int d\boldsymbol{r}_1\,d\boldsymbol{r}_2\,\Phi_{12}\,e^{-\Phi_{12}/k_BT}$$

$$\left\{\int d\boldsymbol{r}_3\left[e^{-(\Phi_{13}+\Phi_{23})/k_BT}-e^{-\Phi_{13}/k_BT}-e^{-\Phi_{23}/k_BT}+1\right\}\right.. \tag{17.26}$$

It defines the first three terms of the equilibrium virial expansion of the internal energy density.

If the Choh-Uhlenbeck kinetic equation is used, then Eqs (17.25), (17.26) contain only the first two terms, describing the internal energy density in the binary collision approximation. This is one of the aspects of the inconsistency of the Choh-Uhlenbeck equation, mentioned above.

If the state of the gas is spatially inhomogeneous, there appear additional terms in the expressions for f_2^{tr} and I^{tr}.

Using the kinetic equation

$$\left[\frac{\partial}{\partial t}+\boldsymbol{v}\cdot\frac{\partial}{\partial\boldsymbol{r}}+\boldsymbol{F}_0\cdot\frac{\partial}{\partial\boldsymbol{p}}\right]f_1 = I^{tr}(\boldsymbol{r},\boldsymbol{p},t) \tag{17.27}$$

it is possible to derive the hydrodynamical equations in the usual way. The expressions for the coefficients of viscosity and of heat conduction coincide with those obtained by Choh and Uhlenbeck [5], but the pressure and the internal energy density contain the first three terms of the virial expansion. Thus, the kinetic equation derived here leads to the hydrodynamic equations containing the effect of triple collisions in the dissipative as well as in the nondissipative quantities.

We recall that in the approximation of binary collisions the viscosity and heat conductivity coefficients, η,κ, depend only on the temperature, not on the density (see Eq. (7.19)). This follows from the fact that, e.g., $\eta\sim\rho\,lv_T$ and the mean free path $l\sim n^{-1}$ and $\rho\sim n$.

In the triple collision approximation, the viscosity and the heat conductivity depend on the density in the form

$$\eta = \eta_0+\varepsilon\eta_1 \qquad \varepsilon = nr_0^3$$
$$\kappa = \kappa_0+\varepsilon\kappa_1 \tag{17.28}$$

where η_0, κ_0 are the corresponding quantities in the pair collision approximation. In the hard-sphere model one obtains the result [5]:

$$\eta_1 = 9.6\ \eta_0 \quad , \quad \kappa_1 = 2.9\ \kappa_0 \ . \tag{17.29}$$

The values of η_0, κ_0 were given in Eq. (7.19).

The method considered here might be generalized to the derivation of kinetic equations describing higher order collisions. However, this generalization does not deserve interest because of the divergence appearing in the next orders of approximation.

In the next section we consider one of the possible methods by which the difficulties of the construction of kinetic equations for dense gases can be overcome.

18. KINETIC EQUATIONS FOR THE SMOOTHED DENSITY OF A GAS

In sections 13 and 17 the kinetic equations for nonideal gases were derived in the binary and ternary collision approximations. The construction of kinetic equations for denser gases, taking into account higher order collisions, cannot be performed according to this scheme, because the collision integrals so obtained diverge (cf. sect. 16).

We saw that in the derivation of the kinetic equations one makes use of the condition of weakening of the fast (short-scale) fluctuations. The boundary between fast and slow fluctuations depends on the approximation considered. It is determined by the size of the corresponding physically infinitesimal intervals, l_ϕ, τ_ϕ (see sect. 16). In the binary collision approximation $\tau_\phi^{(2)} \sim \varepsilon^{\frac{1}{2}} \tau_{\mathbf{rel}}, \tau_\phi \sim \varepsilon^{\frac{1}{2}} l$. From the discussion of sect. 16 it follows that the collision integral in the four-particle collision regime cannot be obtained by the method of perturbation theory in powers of the density; indeed, the quantities $\tau^{(4)} \sim \tau_{\mathbf{rel}}, l^{(4)} \sim l$ and they cannot be considered as negligibly small. As a result, the collision integral depends explicitly on $\tau_{\mathbf{rel}}$ (or l). This leads to an additional dependence of the collision integral on the density.

In other words, in deriving the collision integrals describing four-particle and higher order collisions, it is necessary to take properly into account the contribution of the binary collisions to these higher order expressions.†

We start from the Liouville equation, averaged over a physically infinitesimal volume l_ϕ^3, defined in the binary collision approximation [23, 24] (16.3) and the definition of sect. 2: $l_\phi^3 \sim 1/\varepsilon^{\frac{1}{2}} n$ (remember that $l_\phi \sim \varepsilon^{\frac{1}{2}} l \ll l$). We denote this averaged distribution function by $\tilde{f}_N$.

† We follow here reference [23]. A similar scheme for the ideal gas of hard spheres was developed in the work by Hopfield and Bastin [64].

In order to obtain the equation for this function we proceed as follows. We represent the two-particle function $f_2(x_1, x_2, t)$ in the form:

$$f_2(x_1, x_2, t) = \tilde{f}_2 + f_2^{(1)} \tag{18.1}$$

where $\tilde{f}_2$ is the smoothed distribution function, describing the behaviour of the particles at distances $|\boldsymbol{r}_1 - \boldsymbol{r}_2| \gg l_\phi$

$$f_2^{(1)} = 0 \quad \text{for} \quad |\boldsymbol{r}_1 - \boldsymbol{r}_2| \gg l_\phi \ . \tag{18.2}$$

If the condition of complete weakening of the initial correlations is valid, then, in the zeroth approximation in the density parameter, the function f_2 is expressed in terms of one-particle functions as follows (10.4) :

$$f_2(x_1, x_2, t) = f_1(\boldsymbol{P}_1(-\infty), t)\ f_1(\boldsymbol{P}_2(-\infty), t) \tag{18.3}$$

We recall that $\boldsymbol{P}_{1,2}(-\infty)$ are the initial momenta of two particles colliding at time t .

If, instead, we use only the condition of the weakening of the short-scale fluctuations, with $r_{\text{cor}} \ll l$, $\tau_{\text{cor}} \ll \tau_{\text{rel}}$, then we obtain the more general expression :

$$f_2(x_1, x_2, t) = \tilde{f}_2(\boldsymbol{r}_1, \boldsymbol{P}_1, (-\infty), \boldsymbol{r}_1, \boldsymbol{P}_2(-\infty), t) \ . \tag{18.4}$$

Here $\boldsymbol{r}_2 = \boldsymbol{r}_1$, because in Eq. (18.3) one does not consider spatial variations of the functions f_1 over distances of order $\boldsymbol{r}_0$; the time retardation is also neglected. If these two effects are included (to first order in τ_0/τ_{rel}), one must replace (18.3) by the expressions (15.2), (15.4). Thus, the condition of weakening of the short-scale fluctuation is expressed by the more general relation

$$f_2(x_1, x_2, t) = \left(1 - \frac{\boldsymbol{r}_{12}}{2} \cdot \frac{\partial}{\partial \boldsymbol{r}_1}\right) \tilde{f}_2(\boldsymbol{r}_1, \boldsymbol{P}_1(-\infty), \boldsymbol{r}_1, \boldsymbol{P}_2(-\infty), t)$$
$$- \left(\frac{\partial}{\partial t} + \frac{\boldsymbol{v}_1 + \boldsymbol{v}_2}{2} \cdot \frac{\partial}{\partial \boldsymbol{r}_1}\right) \int_0^\infty d\tau\, \tau \frac{d}{d\tau} \tilde{f}_2\ (\boldsymbol{r}_1, \boldsymbol{P}_1(-\tau), \boldsymbol{r}_1, \boldsymbol{P}_2(-\tau), t) \ . \tag{18.5}$$

If we limit ourselves to Eq. (18.4), it will be seen that the effect of the binary collisions in the equation for the smoothed function $\tilde{f}_N$ and in the corresponding equations for $\tilde{f}_1, \tilde{f}_2, \tilde{f}_3, \ldots$ appears only in the dissipative contribution. The use of (18.5) to first order in τ_0/τ_{rel}, r_0/l includes also the correction to the nondissipative contributions.

For simplicity we shall use the approximation (18.4). The nondissipative contributions of the binary collisions can be added to the final results.

For $|\boldsymbol{r}_1 - \boldsymbol{r}_2| < l_\phi$ the function (18.4) is a rapidly varying function of $\boldsymbol{r}_1, \boldsymbol{r}_2$ and corresponds to the function $f_2^{(1)}$ in (18.1). For $|\boldsymbol{r}_1 - \boldsymbol{r}_2| > l_\phi$ in (18.4) $\boldsymbol{P}_{1,2} = \boldsymbol{p}_{1,2}$ and $f_2(x_1, x_2, t) = \tilde{f}_2(x_1, x_2, t)$, which follows already from the definition (18.1).

In the weak-coupling approximation the moments $\boldsymbol{P}_{1,2}(-\tau)$ are defined by Eq. (13.7). Using this expression we find from (18.4) the following weak-coupling approximation form for f_2:

$$f_2(x_1,x_2,t) = \tilde{f}_2(x_1,x_2,t) + \int_0^\infty d\tau \, \frac{\partial \phi_{12}(|\boldsymbol{r}_1 - \boldsymbol{r}_2 - (\boldsymbol{v}_1 - \boldsymbol{v}_2)\tau|)}{\partial \boldsymbol{r}_1} \cdot \left(\frac{\partial}{\partial \boldsymbol{p}_1} - \frac{\partial}{\partial \boldsymbol{p}_2}\right) \tilde{f}_2(x_1,x_2,t). \tag{18.6}$$

The second term vanishes for $|\boldsymbol{r}_1 - \boldsymbol{r}_2| > l_\phi$ and corresponds to $f_2^{(1)}$ in (18.1)

From the structure of the Liouville equation (1.11)

$$\left(\hat{L}_{x_1 \ldots x_N} - \sum_{1 \leqslant i < j \leqslant N} \hat{\theta}_{ij}\right) f_N = 0, \tag{18.7}$$

follows that in the pair approximation in each term $\hat{\theta}_{ij} f_N$, the short-scale motion must be taken into account only for the particles i, j. We introduce the following notation:

$$f_N^{(i,j)} = \tilde{f}_N(x_1, \ldots, \boldsymbol{r}_i, \boldsymbol{P}_i, \ldots, \boldsymbol{r}_j, \boldsymbol{P}_j, \ldots, x_N, t). \tag{18.8}$$

One can set here $\boldsymbol{r}_i = \boldsymbol{r}_j$, as the nonuniformity on short scales can be neglected.

Taking into account the fact that

$$\overrightarrow{\hat{\theta}_{ij} f_N} = \tilde{\hat{\theta}}_{ij} \tilde{f}_N + \overrightarrow{\hat{\theta}_{ij} f_N^{(i,j)}} \tag{18.9}$$

we obtain the Liouville equation for the smoothed distribution function

$$\hat{L}^{(0)}_{x_1 \ldots x_N} \tilde{f}_N - \sum_{1 \leqslant i < j \leqslant N} \tilde{\hat{\theta}}_{ij} \tilde{f}_N = \sum_{1 \leqslant i < j \leqslant N} \overrightarrow{\hat{\theta}_{ij} f_N^{(i,j)}}. \tag{18.10}$$

Henceforth, the tilde ~ will be deleted over $\hat{\theta}_{ij}$ and over the right-hand side in (18.10) whenever this does not lead to confusion.

On the basis of Eq. (18.10) one can construct the hierarchy of equations for the smoothed functions $\tilde{f}_1 \equiv \tilde{f}_1, \tilde{f}_2, \tilde{f}_3, \ldots$. From Eq. (18.10) we obtain for the one-particle function:

$$\left(\frac{\partial}{\partial t} + \boldsymbol{v}_1 \cdot \frac{\partial}{\partial \boldsymbol{r}_1} + \boldsymbol{F}_0 \cdot \frac{\partial}{\partial \boldsymbol{p}_1}\right) f_1 = \frac{N-1}{V} \int dx_2 \, \hat{\theta}_{12} \left[\tilde{f}_2(x_1,x_2,t) + f_2^{(1,2)}(x_1,x_2,t)\right] \tag{18.11}$$

where, from (18.8),

$$f_2^{(1,2)}(x_1,x_2,t) = \tilde{f}_2(\boldsymbol{r}_1, \boldsymbol{P}_1(-\infty), \boldsymbol{r}_1, \boldsymbol{P}_2(-\infty), t). \tag{18.12}$$

We rewrite the second equation of the hierarchy as

$$\hat{L}^{0}_{x_1,x_2}\tilde{f}_2 - \hat{\theta}_{12}\left(\tilde{f}_2 + f_2^{(1,2)}\right)$$

$$= \frac{N-2}{V}\int dx_3\left[\hat{\theta}_{13}\left(\tilde{f}_3 + f_3^{(1,3)}\right) + \hat{\theta}_{23}\left(\tilde{f}_3 + f_3^{(2,3)}\right)\right] . \qquad (18.13)$$

Clearly, this hierarchy for the smoothed functions is not closed. Hence the problem of its termination arises again: one needs approximation methods for obtaining closed sets for a finite number of smoothed distribution functions.

We introduce two-particle correlation functions corresponding to $\tilde{f}_2, f_2^{(1,2)}$. Using (18.4) we get

$$\tilde{f}_2(x_1,x_2,t) = f_1(x_1,t)\ f_1(x_2,t) + \tilde{g}_2(x_1,x_2,t) \qquad (18.14)$$

$$f_2^{(1,2)}(x_1,x_2,t) = f_1(\boldsymbol{r}_1,\boldsymbol{P}_1(-\infty),t)\ f_1(\boldsymbol{r}_1,\boldsymbol{P}_2(-\infty),t)$$

$$+ \tilde{g}_2(\boldsymbol{r}_1,\boldsymbol{P}_1(-\infty),\boldsymbol{r}_1,\boldsymbol{P}_2(-\infty),t) . \qquad (18.15)$$

If the correlation functions in (18.14), (18.15) were zero, the first equation of the hierarchy,(18.10), is closed and corresponds to the Boltzmann equation with the collision integral (10.5) or (15.6).

Thus, the same cut-off of the hierarchy for the smoothed distribution functions, corresponding to the complete neglect of $\tilde{g}_2, \tilde{g}_3, \ldots,$ leads to the Boltzmann equation for the ideal gas. The latter takes into account the dissipation through binary collisions.

If instead of (18.4) one had used the more general expression (18.5), we would have obtained in zeroth approximation the Boltzmann equation for the nonideal gas, with the collision integral (15.5).

We consider now higher approximations for the correlation functions. On the small scales, the correlation functions $\tilde{g}_2$ can be neglected in (18.15). Indeed, on such scales (or order r_0) the main role is played by the correlations due to pair collisions, but these are taken into account by the first term in (18.15).

With these assumptions the first equation of the hierarchy (18.11), using (18.14) can be rewritten as

$$\left(\frac{\partial}{\partial t} + \boldsymbol{v}_1\cdot\frac{\partial}{\partial \boldsymbol{r}_1} + \boldsymbol{F}_0\cdot\frac{\partial}{\partial \boldsymbol{p}_1}\right) f_1(x_1,t) = I(x_1,t)$$

$$\equiv I_{\mathrm{B}}(x_1,t) + \hat{I}(x_1,t) . \qquad (18.16)$$

Here I is the corresponding collision integral. It consists of two terms: I_{B} is the Boltzmann collision integral (10.5) or the more general collision integral (15.5) - (15.8), and $\tilde{I}$ the part of the collision integral defined by the long-range correlations :

$$\tilde{I} = \frac{N-1}{V} \int dx_2 \frac{\partial \Phi_{12}}{\partial \boldsymbol{r}_1} \cdot \frac{\partial}{\partial \boldsymbol{p}_1} \tilde{g}_2(x_1, x_2, t) .$$

Here $(N-1)/V$ can be replaced by n. Through this term the interactions of higher order than the binary ones can be introduced into the kinetic equations.

We come back to Eqs (18.13) for $\tilde{f}_2$ and express the distributions $f_3^{(1,3)}$, $f_3^{(2,3)}$ through correlation functions. These relations are analogous to (2.5):

$$\begin{aligned} f_3^{(1,3)}(x_1,x_2,x_3,t) = {} & f_1(\boldsymbol{r}_1, \boldsymbol{P}_1, t) f_1(x_2,t)\, f_1(\boldsymbol{r}_1, \boldsymbol{P}_3, t) \\ & + f_1(\boldsymbol{r}_1, \boldsymbol{P}_1, t)\, \tilde{g}_2(x_2, \boldsymbol{r}_1, \boldsymbol{P}_3, t) \\ & + f_1(x_2, t)\, \tilde{g}_2(\boldsymbol{r}_1, \boldsymbol{P}_1, \boldsymbol{r}_1, \boldsymbol{P}_3, t) \\ & + f_1(\boldsymbol{r}_1, \boldsymbol{P}_3, \ )\, \tilde{g}_2(\boldsymbol{r}_1, \boldsymbol{P}_1, \boldsymbol{r}_1, \boldsymbol{P}_3, t) \\ & + \tilde{g}_3(\boldsymbol{r}_1, \boldsymbol{P}_1, x_2, \boldsymbol{r}_1, \boldsymbol{P}_3, t) \end{aligned} \tag{18.18}$$

The expression for the function $f_3^{(2,3)}$ follows from (18.18) by the permutation of the indices 1 and 2. In Eq. (18.18) the third and the last terms can be neglected, as the long-scale correlations are unimportant at short distances. After these simplifications. we substitute (18.18) and the corresponding expression of $f_3^{(2,3)}$ into the right-hand side of (18.13). We introduce the following symbol for the linearized Boltzmann collision integral (10.5) :

$$\begin{aligned} \delta \hat{I}_{x_1} \left\{ \delta f(x_1, t) \right\} = - n \int dx_2\, \hat{\theta}_{12} \Big\{ & f_1 [\boldsymbol{r}_1, \boldsymbol{P}_1(-\infty), t]\, \delta f [\boldsymbol{r}_1, \boldsymbol{P}_2(-\infty), t] \\ & + \delta f [\boldsymbol{r}_1, \boldsymbol{P}_1(-\infty), t]\, f_1 [\boldsymbol{r}_1, \boldsymbol{P}_2(-\infty), t] \Big\} \end{aligned} \tag{18.19}$$

The minus sign introduced in this definition is convenient for further use.

Consider the result of the action of the operator $\delta\hat{I}$ on the correlation function $\tilde{g}_2$. For instance :

$$\begin{aligned} & \delta \hat{I}_{x_1} \left\{ \tilde{g}_2(x_1, x_2, t) \right\} \\ & = -n \int dx_3\, \hat{\theta}_{13} \Big\{ f_1(\boldsymbol{r}_1, \boldsymbol{P}_1(-\infty), t)\, \tilde{g}_2(x_2, \boldsymbol{r}_1, \boldsymbol{P}_3(-\infty), t) \\ & + \tilde{g}_2(\boldsymbol{r}_1, \boldsymbol{P}_1(-\infty), x_2, t)\, f_1(\boldsymbol{r}_1, \boldsymbol{P}_3(-\infty), t) \Big\} . \end{aligned} \tag{18.20}$$

An analogous definition holds for the expression $\delta \hat{I}_{x_2} \{ \tilde{g}_2(x_1, x_2, t) \}$.

Using these notations we rewrite (18.13) in the form

$$L^{0}_{x_1 x_2}\tilde{f}_2 + \left(\delta\hat{I}_{x_1} + \delta\hat{I}_{x_2}\right)\tilde{g}_2 = \hat{\tilde{\theta}}_{12}\tilde{f}_2 + \overbrace{\hat{\theta}_{12} f_1(\boldsymbol{r}_1, \boldsymbol{P}_1, t) f_1(\boldsymbol{r}_1, \boldsymbol{P}_2, t)}$$

$$-\frac{1}{N}\left[I_B(x_1,t) f_1(x_2,t) + I_B(x_2,t) f_1(x_1,t)\right]$$

$$+\frac{N-2}{V}\int (\hat{\theta}_{13} + \hat{\theta}_{23})\tilde{f}_3$$

$$+\left[I_B(x_1,t) f_1(x_2,t) + I_B(x_2,t) f_1(x_1,t)\right. . \tag{18.21}$$

The Boltzmann collision integral appearing on the right-hand side is

$$I_B(x_1,t) = \frac{N-1}{V}\int dx_2\, \hat{\theta}_{12} f_1(\boldsymbol{r}_1, \boldsymbol{P}_1(-\infty), t)\, f_1(\boldsymbol{r}_1, \boldsymbol{P}_2(-\infty), t).$$

In (18.21) we separated from the last term a part proportional to $1/(N-1) \approx 1/N$. The meaning of such a separation will become clear below. We did not take the limit $N\to\infty$, $V\to\infty$ in order to leave open the possibility of studying the fluctuations in systems with a finite number of particles. One can however replace, of course, $N-1$ by N, as $N \gg 1$.

If Eq. (18.21) is multiplied by $1/V$ and integrated over x_2, it coincides with the kinetic equation (18.16). In order to show this one must use the definition (18.20) and the property (2.8) of the correlation functions.

We continue the transformation of Eq. (18.21). We substitute into the left-hand side the expression $f_1 f_1 + \tilde{g}_2$ for $\tilde{f}_2$ and eliminate the derivatives of f_1 by means of the kinetic equation (18.16). We obtain

$$\left(\hat{L}^0_{x_1,x_2} + \delta\hat{I}_{x_1} + \delta\hat{I}_{x_2}\right)\tilde{g}_2(1,2)$$

$$= \left[\hat{\theta}_{12}\tilde{f}_2(1,2) - \frac{1}{V}\int dx_3\left\{\hat{\theta}_{13}\tilde{f}_2(1,3) f_1(2) + \hat{\theta}_{23}\tilde{f}_2(2,3) f_1(1)\right]\right\}$$

$$+\hat{\theta}_{12} f_1(\boldsymbol{r}_1, \boldsymbol{P}_1, t)\, f_1(\boldsymbol{r}_1, \boldsymbol{P}_2, t)$$

$$-\frac{1}{N}\left[I_B(x_1,t) f_1(2) + I_B(x_2,t) f_1(1)\right]$$

$$+\frac{N-2}{V}\int dx_3\left\{\hat{\theta}_{13}\left[\tilde{f}_3(1,2,3) - \tilde{f}_2(1,3) f_1(2)\right]\right.$$

$$\left.+\hat{\theta}_{23}\left[\tilde{f}_3(1,2,3) - \tilde{f}_2(2,3) f_1(1)\right]\right\} \tag{18.22}$$

The second term on the right-hand side appeared because in front of the integrals containing $\tilde{f}_2$ there is a factor $(N-2)/V$, whereas in (18.16) the factor is $(N-1)/V$.

We eliminate the functions $\tilde{f}_2$, $\tilde{f}_3$ by means of Eqs (18.14) and the corresponding expressions for $\tilde{f}_3$:

$$\tilde{f}_3(1,2,3) = f_1(1)\,f_1(2)\,f_1(3) + f_1(1)\,\tilde{g}_2(2,3)$$
$$+ f_1(2)\,\tilde{g}_2(1,3) + f_1(3)\,\tilde{g}_2(1,2) + \tilde{g}_3(1,2,3). \quad (18.25)$$

The bracketed expressions in the last term of (18.22) become

$$\tilde{f}_3(1,2,3) - \tilde{f}_2(1,3)f_1(2) = f_1(1)\,\tilde{g}_2(2,3) + f_1(3)\,\tilde{g}_2(1,2) + \tilde{g}_3(1,2,3)$$
$$\tilde{f}_3(1,2,3) - \tilde{f}_2(2,3)f_1(1) = f_1(2)\,\tilde{g}_2(1,3) + f_1(3)\,\tilde{g}_2(1,2) + \tilde{g}_3(1,2,3)$$
$$(18.24)$$

We neglect the terms containing the function $f_1(3)$ on the left-hand side and use the relation (2.10) for the average force. Hence, on the left-hand side, instead of $\hat{L}_0$, appears the operator $\hat{L}$. It is defined by Eq. (1.8) in which the force $\boldsymbol{F}_0$ is replaced by $\boldsymbol{F}$. As a result, we obtain the following equation for $\tilde{g}_2$

$$\left(\hat{L}_{x_1\,x_2} + \delta\hat{I}_{x_1} + \delta\hat{I}_{x_2}\right)\tilde{g}_2(1,2) = \hat{\theta}_{12}[f_1(1)\,f_1(2) + g_2(1,2)]$$
$$- \frac{1}{V}\int dx_3 \left\{\hat{\theta}_{13}\left[f_1(1)\,f_1(3) + \tilde{g}_2(1,3)\right] f_1(2) + \right.$$
$$\left. + \hat{\theta}_{23}\left[f_1(2)\,f_1(3) + \tilde{g}_2(2,3)\right] f_1(1)\right\}$$
$$+ \hat{\theta}_{12}\, f_1(\boldsymbol{P}_1,t)\, f_1(\boldsymbol{P}_2,t) - \frac{1}{N}\left[I_B(x_1,t)\,f_1(2) + I_B(x_2,t)\,f_1(1)\right]$$
$$+ n\int dx_3 \left\{\hat{\theta}_{13}\left[f_1(1)\,\tilde{g}_2(2,3) + \tilde{g}_3(1,2,3)\right] + \right.$$
$$\left. + \hat{\theta}_{23}\left[f_1(2)\,\tilde{g}_2(1,3) + \tilde{g}_3(1,2,3)\right]\right\}. \quad (18.25)$$

Let us discuss the role of the various terms on the right-hand side of this equation. The last term is proportional to n: it describes triple and higher order interactions

The first four terms describe binary collision contributions. The latter can be split in turn into two parts. The first two terms define the contributions of the interactions at distances $r_{12} > l_\phi$, and the next two those at distances $r_{12} < l_\phi$. The second and fourth terms, proportional respectively to $1/V$ and to $1/N$, ensure the existence of solutions of Eq. (18.25) for which

$$\frac{1}{V}\int dx_2\, \tilde{g}_2(x_1,x_2,t) = \frac{1}{V}\int dx_1\, \tilde{g}_2(x_1,x_2,t) = 0 .$$

The fulfilment of these relations is necessary for the consistency with the definition of the correlation function $\tilde{g}_2$ [cf. the conditions (2.8)].

The contribution of the ternary and higher-order interactions enters the kinetic equation (18.16) through the integral $\tilde{I}$. The latter is defined through $\tilde{g}_2$, which in turn is determined through Eq. (18.25).

We now compare Eq. (18.25) to the second equation of the BBGKY hierarchy

(1.20). For convenience we write, instead of (1.20), an equation for the correlation function $g_2 \equiv f_2 - f_1 f_1$. From Eqs (1.20), (1.19), taking into account (2.4) and (2.5) we obtain

$$\left(\hat{L}_{x_1,x_2} - \hat{\theta}_{12}\right) g_2(1,2) = \hat{\theta}_{12}\, f_1(1)\, f_1(2) + n\int dx_3 \Big\{\hat{\theta}_{13}\big[f_1(1)\; g_2(2,3) + g_3(1,2,3)\big] + \hat{\theta}_{23}\big[f_1(2)\, g_2(1,3) + g_3(1,2,3)\big]\Big\}. \qquad (18.26)$$

Here, unlike (18.25), there are no terms proportional to $1/V$.

In the binary collision approximation the equation for the function g_2 [(2.19)] can be written as

$$\left(\hat{L}_{x_1,x_2} - \hat{\theta}_{12}\right) g_2(1,2) = \hat{\theta}_{12}\, f_1(1)\, f_1(2). \qquad (18.27)$$

Comparing Eqs (18.26) and (18.27), we see that the contribution of ternary and higher order interactions is described by the second term — proportional to n — on the right-hand side of Eq. (18.26). This term has the same form as the corresponding one in (18.25).

It was said in section 16 that the density expansion of the BBGKY equations leads to kinetic equations with divergent integrals. This statement changes, however, if the density expansion is applied to the solution of the hierarchy of equations for the smoothed distribution functions $\tilde{f}_1, \tilde{f}_2, \ldots$

As in the Eqs (18.25) and (18.26) the terms describing the contributions of ternary and higher-order interactions coincide, the main difference between these equations consists in the replacement of the operator

$$\hat{L}_{x_1,x_2} - \hat{\theta}_{12} \qquad (18.28)$$

in (18.26) by the operator

$$\hat{L}_{x_1,x_2} - \hat{\theta}_{12} + \delta\hat{I}_{x_1} + \delta\hat{I}_{x_2} \qquad (18.29)$$

in (18.25).

In solving Eq. (18.25). the solution of the homogeneous equation can be neglected: because of the presence of the terms $\delta\hat{I}_{x_1}$, $\delta\hat{I}_{x_2}$, the initial correlation $\tilde{g}_2$ decays in a time of order $\tau^{(2)}_{\mathbf{rel}}$, and the characteristic times of the triple and four-body collisions are of the order $\tau^{(2)}_{\mathbf{rel}}/\varepsilon$, $\tau^{(2)}_{\mathbf{rel}}/\varepsilon^2$, respectively.

In the solution of the inhomogeneous equation (18.25), the presence of the terms $\left(\delta\hat{I}_{x_1} + \delta\hat{I}_{x_2}\right)\tilde{g}_2 \sim \tilde{g}_2/\tau^{(2)}_{\mathbf{rel}}$ leads to a cut-off of the collision integrals at a time $t \sim \tau^{(2)}_{\mathbf{rel}}$; i.e. at a time of the order of the mean free time for binary collisions.

For instance, considering triple collisions in Eq. (17.9) we are led to

the change

$$\int_0^\infty d\tau \; S_{-\tau}^{(2)}(1,2) \ldots \quad \longrightarrow \quad \int_0^\infty d\tau \; e^{-\tau/\tau_{\mathbf{rel}}} \, S_{-\tau}^{(2)}(1,2) \ldots . \tag{18.30}$$

In this way, the divergence in the collision integrals describing higher order collisions are suppressed as a result of the dissipative contribution due to the pair collisions. An analogous result can be obtained by means of the diagram technique. One must then sum the most divergent diagrams arising in the density expansion [22 , 24].

The appearance of the new cut-off at $t \sim \tau_{\mathbf{rel}}^{(2)}$ leads to an additional density dependence in the collision integrals and in the corresponding transport coefficients.

For instance, for the viscosity η, instead of the alleged virial expansion in powers of the density parameter $\varepsilon = nr_0^3$:

$$\eta = \eta_0(T) + \varepsilon\eta_1(T) + \varepsilon^2\eta_2(T) + \ldots \tag{18.31}$$

the first three terms in the series are

$$\eta = \eta_0(T) + \varepsilon\eta_1(T) + \varepsilon^2 \ln(1/\varepsilon)\, \eta'(T) . \tag{18.32}$$

In (18.31), (18.32), the coefficients η_0, η_1 are defined respectively by binary and triple collisions. The next term in the virial series cannot be found, because of the divergence of the integral describing four-body collisions. The divergence has a logarithmic character $\ln(\tau/\tau_0)$. Hence, in presence of the cut-off (18.30), $\ln(\tau/\tau_0)$ can be approximated by

$$\ln(\tau_{\mathbf{rel}}^{(2)}/\tau_0) = \ln(1/\varepsilon) \tag{18.33}$$

As a result, the third term in (18.32), describing the contribution of four-body collisions. is proportional to $\varepsilon^2 \ln(1/\varepsilon)$ and hence depends non-analytically on the density parameter. The existing experimental data on the density dependence of the viscosity do not yet allow us to distinguish the logarithmic term [55].

19. ON THE STATISTICAL DESCRIPTION OF NON-EQUILIBRIUM PROCESSES IN DENSE GASES

We considered the kinetic equations for nonideal gases in the approximations of binary and triple collisions. As shown in section 18, it is possible to derive kinetic equations also for denser gases, where four-body and higher-order collisions play an important role. Such a method of successive approximations is, however, not very effective. We need here some other scheme, analogous to the one used in the equilibrium statistical theory of dense gases and liquids (Sect. 9).

For thermodynamic functions there exists a density expansion (Sect. 8): the virial expansion. But in practice only the first few terms of this series are used. The calculation of the thermodynamic functions of dense gases and of liquids

proceeds either via numerical methods (the method of molecular dynamics or the Monte-Carlo method) or through the (analytical or numerical) solution of model integral equations for the pair correlation function g_2. In section 9 we considered several such equations. All of them provide the exact value of the first three virial coefficients and give approximate expressions for the higher ones.

An analogous procedure is possible also for the construction of the kinetic theory of dense gases. Instead of a single kinetic equation for the function f_1, the description of non-equilibrium processes in such systems can be based on a set of two equations for f_1, g_2 or for f_1, f_2. Let us consider one example of such a set.

We saw that Eq. (17.11) for the function f_2 is defined through a sum of contributions of the binary and the triple collisions, with due account for the retardation. The contribution of the binary collisions is described by the first term in the right-hand side of (17.11): it consists in turn of two terms [see (17.5)]. The first one is of zeroth order in the density and describes in the collision integral the dissipative contribution, in the pair-collision approximation. The second term defines the contribution of the pair-collisions to the non-dissipative functions, such as the internal energy, under the $(\partial/\partial t)$ sign. Formally, this term might be considered as a contribution to f_2, of first order in the density. But, as shown in section 17, the consideration of the formal density expansion of f_2 leads to an inconsistent description of the dissipative and nondissipative processes.

Taking this fact into account we consider, for the generalization of Eq. (17.11), the expression $f_2^{\mathbf{bc}}$ given by (17.5) as a zeroth approximation for f_2. It describes the complete contribution of the binary collisions to the kinetic equation. The first approximation is described by the remaining terms in (17.11): they define the complete contribution of the triple collisions to the kinetic equation.

Let us use for the distribution functions the analogue of Kirkwood's superposition approximation in the form

$$f_3(1,2,3) = f_3^{\mathbf{tr}} \frac{f_2(1,2)\, f_2(1,3)\, f_2(2,3)}{f_2^{\mathbf{bc}}(1,2)\, f_2^{\mathbf{bc}}(1,3)\, f_2^{\mathbf{bc}}(2,3)} \tag{19.1}$$

The functions $f_2^{\mathbf{bc}}, f_3^{\mathbf{tr}}$ are defined by Eqs (17.5), (17.7).

In first approximation, when [see (17.5)]

$$f_2 = f_2^{\mathbf{bc}} = S_{-\infty}^{(2)} f_1 f_1 - \frac{\partial}{\partial t} \int_0^\infty d\tau\ \tau \frac{d}{d\tau} S_{-\tau}^{(2)} f_1 f_1 \equiv f_2^0 + f_2^1 , \tag{19.2}$$

we find from (19.1), (19.2), (17.7)

$$f_3 = f_3^{\mathbf{tr}} = S_{-\infty}^{(3)} f_1 f_1 f_1 - \frac{\partial}{\partial t} \int_0^\infty d\tau\ \tau \frac{d}{d\tau} S_{-\tau}^{(3)} f_1 f_1 f_1 \equiv f_3^0 + f_3^1 . \tag{19.2}$$

Consider the equilibrium state. As the position distribution functions are

$$f_2^{\mathrm{bc}}(\mathbf{r}_1, \mathbf{r}_2) = \exp(-\Phi_{12}/k_B T)$$
$$f_3^{\mathrm{tr}}(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3) = \exp\left[-(\Phi_{12}+\Phi_{13}+\Phi_{23})/k_B T\right] \tag{19.4}$$

we find for the three-particle distribution:

$$f_3(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3) = f_2(\mathbf{r}_1, \mathbf{r}_2)\, f_2(\mathbf{r}_1, \mathbf{r}_3)\, f_2(\mathbf{r}_2, \mathbf{r}_3) \tag{19.5}$$

which coincides with Kirkwood's superposition approximation (19.1).

In order to define f_2, equation (17.11) which defines f_2 in the triple-collision approximation, can be changed into an integral equation

$$\begin{aligned}
f_2(1,2) = {} & f_2^0(1,2) + f_2^1(1,2) + n\int_0^\infty d\tau \int dx_3\; S_{-\tau}^{(2)}(1,2)\,\cdot \\
& \cdot\left[(\hat\theta_{13}+\hat\theta_{23})\, f_3(1,2,3) - \hat\theta_{13}\, f_2(1,3)\, f_1(2) - \hat\theta_{23}\, f_2(2,3)\, f_1(1)\right] \\
& - n\,\frac{\partial}{\partial t}\int_0^\infty d\tau \int dx_3\; \tau\, S_{-\tau}^{(2)}(1,2)\left\{(\hat\theta_{13}+\hat\theta_{23})\left[f_3(1,2,3) - f_3^1(1,2,3)\right]\right. \\
& - S_{-\infty}^{(2)}(1,2)\left[\hat\theta_{13}\left\{f_2(1,3) - f_2^1(1,3)\right\} f_1(2) + \right. \\
& \left.\left. + \hat\theta_{23}\left\{f_2(2,3) - f_2^1(2,3)\right\} f_1(1)\right]\right\} .
\end{aligned} \tag{19.6}$$

The functions f_2^0, f_2^1, f_3^1, entering this integral equation, are defined by Eqs. (19.2), (19.3) and f_3 by (19.1).

In the triple collision approximation one must set under the integrals in Eq. (19.6): $f_2 = f_2^{\mathrm{bc}} \equiv f_2^0 + f_2^1$ and $f_3 = f_3^{\mathrm{tr}} \equiv f_3^0 + f_3^1$, ... Then (19.6) reduces to (17.11).

Equations (17.2), (19.6) are a closed set of equations for the functions f_1, f_2. In equilibrium, Eq. (19.6) reduces to the corresponding equation obtained from the equilibrium hierarchy in Kirkwood's superposition approximation (sect. 18).

One can use instead of (19.6) other types of approximate integral equations for $f_2(x_1, x_2, t)$, which reduce in equilibrium to the Percus-Yevick equations (9.2) (9.3) or to Eq. (9.6).

20 GENERALIZED KINETIC EQUATION

In section 16 we noted already that in recent years there have appeared a series of papers in which one tries, by various methods, to obtain generalized kinetic equations describing correlations of arbitrary order. A simple and clear solution of this problem appears in the work of Zubarev and Novikov [52]. The essence of their work is as follows.

We express the collision integral through the phase-space distribution function $F_N = V^N f_N$. Note the normalization

$$\frac{1}{V^N} \int dx_1 , \ldots , dx_N \; F_N = 1 \; .$$

For spatially homogeneous systems we obtain from (2.13)

$$\frac{\partial f_1}{\partial t} = I(\boldsymbol{p}_1 , t) = n \frac{1}{V^{N-2}} \int dx_2 , \ldots , dx_N \frac{\partial \Phi_{12}}{\partial \boldsymbol{r}_1} \cdot \frac{\partial F_N}{\partial \boldsymbol{p}_1} \; . \qquad (20.1)$$

For the function F_N, the authors of reference [52] use the Liouville equation. Solving this equation with the condition of weakening of all the initial correlations, one can express F_N in terms of one-particle distribution functions. Substituting this expression into (20.1) one obtains a closed equation for the function f_1 : the generalized kinetic equation.

However, the use of the Liouville equation for F_N leads to some difficulties in the passage to the usual (Boltzmann) form of the collision integral and in the derivation of the conservation laws. These difficulties can be avoided if one uses for F_N, instead of the Liouville equation, the equation

$$\left(\frac{\partial}{\partial t} + \hat{H}_N\right) F_N = \left[\frac{\partial}{\partial t} + \sum_{i=1}^{N} \left(\boldsymbol{v}_i \cdot \frac{\partial}{\partial \boldsymbol{r}_i} + \boldsymbol{F} \cdot \frac{\partial}{\partial \boldsymbol{p}_i}\right)\right] \prod_{j=1}^{N} f_1 (x_j , t) \; . \qquad (20.2)$$

This equation has the same structure as Eq. (2.18) for f_2 in the binary collision approximation, or of Eq. (17.4) with the right-hand side of (17.6) in the ternary collision approximation.

Clearly, Eq. (20.2) is not the Liouville equation. It is the N^{th} equation of the BBGKY-hierarchy in the approximation in which the collisions of $N+1$ particles are neglected. We assume $N \gg 1$.

Under the assumption of the complete weakening of the initial correlations, the solution of Eq. (20.2) for a homogeneous gas can be written as [25]

$$F_N = \prod_{i=1}^{N} f_1(x_i , t) + \int_0^{\tau} d\tau' \frac{d}{d\tau'} \left(S^{(N)}_{-\tau'} \right) \prod_{i=1}^{N} f_1(x_i , t-\tau') . \qquad (20.3)$$

Substituting (20.3) into (20.1) we obtain the collision integral of the generalized kinetic equation.

The previously considered kinetic equations follow from the generalized kinetic equation by using the virial expansion of the non-equilibrium distribution functions. This expansion was obtained in the work of Cohen [22] and other authors.

The collision integral obtained in this way possesses the properties (11.5) for $\phi = \boldsymbol{p}_1$. For $\phi = p_1^2/2m$, one obtains from (20.1), (20.3):

$$n \int dx_1 \, \frac{p_1^2}{2m} \, I(x_1, t) = - \frac{n^2}{2} \frac{\partial}{\partial t} \frac{1}{V^{N-1}} \int dx_1 \, , \ldots , \, dx_N \, \Phi_{12} \, F_N \, . \qquad (20.4)$$

Hence, the internal energy density has the following form:

$$U = n \frac{1}{V} \int dx_1 \, \frac{p_1^2}{2m} \, f_1 + \frac{n^2}{2} \frac{1}{V^{N-1}} \int dx_1 \, , \ldots , \, dx_N \, \Phi_{12} \, F_N \, . \qquad (20.5)$$

In equilibrium, Eq. (20.3) reduces to the Gibbs distribution function for N particles.

The generalized kinetic equations have not yet been used for the solution of concrete problems.

CHAPTER 4

Kinetic Theory of Fluctuations in Gases

21. EQUATION FOR THE SMOOTHED DISTRIBUTION FUNCTIONS IN THE POLARIZATION APPROXIMATION

Let us consider again a gas which is sufficiently dilute for the binary collision approximation to be applicable. We recall that for the derivation of the Boltzmann equation the condition of complete weakening of initial correlations is used. Actually, as we noted in section 16, the weakening affects of the short-scale correlations with $\tau_{cor} \sim \varepsilon^{\frac{1}{2}} \tau_{rel}$, $r_{cor} \sim \varepsilon^{\frac{1}{2}} l$. Thus, in the derivation of the Boltzmann equation, it is essential to assume that the long-range fluctuations (with $\tau_{cor} \gtrsim \varepsilon^{\frac{1}{2}} \tau_{rel}$, $r_{cor} \gtrsim \varepsilon^{\frac{1}{2}} l$) do not play any role in kinetic theory. Only at that price is it possible to obtain a kinetic equation, i.e. a closed equation for f_1.

The long-scale fluctuations do not have time to decay during the relaxation time of f_1. Therefore, when such fluctuations are present, one does not obtain from the BBGKY hierarchy (or from the Liouville equation) the Boltzmann equation, but rather a set of equations for the function f_1 and for the long-range correlations $\tilde{g}_2, \ldots$. This set can be replaced approximately by a Langevin equation for the random function f_1, i.e. by a Boltzmann equation with a stochastic source $y(x_1, t)$.

In this connection there arises naturally the question of the definition of the statistical characteristics of the random source. For the states of the gas not far from equilibrium, this question was first solved by Kadomtsev [25]. In his work the correlation functions of the random source y were calculated by using methods analogous to those developed for the Boltzmann equation itself.

In the work of Gorkov, Dzyaloshinsky and Pitaevsky [38] the methods of Landau and Lifzhitz [56,59] and of Rytov [57,58] were used for the construction of the kinetic theory of fluctuations.

The direct generalization of the work of Kadomtsev [25] to cover non-

equilibrium, but stationary, states of the gas was obtained in the work of Kogan and Shulman [39]. In the work of Gantsevich, Gurevich and Katilius [40, 41] of Jigulev [43], of Sadovnikov [44], and of Kogan [45], the theory of equilibrium states and of stationary non-equilibrium states was developed on the basis of the BBGKY hierarchy or of the corresponding Green functions hierarchy.†

Here we shall proceed to the calculations of the fluctuations from the hierarchy of equations for the smoothed functions f_1, $\tilde{f}_2$, $\tilde{f}_3$, (see sect. 18) and from the corresponding hierarchy for the phase densities $\tilde{N}(x,t)$ [23, 24].

Let us return to Eqs (18.11), (18.13), the first two equations of this heirarchy. For short distances (of order r_0) the main role is played by the correlations due to binary collisions. Therefore, in Eq. (18.15) the contribution of the term $\tilde{g}_2(\boldsymbol{r}_1, \boldsymbol{P}_1(-\infty), \boldsymbol{r}_1, \boldsymbol{P}_2(-\infty))$ can be neglected and one finds for f_1 Eq. (18.16):

$$\left(\frac{\partial}{\partial t} + \boldsymbol{V}_1 \cdot \frac{\partial}{\partial \boldsymbol{r}_1} + \boldsymbol{F} \cdot \frac{\partial}{\partial \boldsymbol{p}_1}\right) f_1(x_1, t) = I(x_1, t) \equiv I_B + \tilde{I} \tag{21.1}$$

Here I_B is the Boltzmann collision integral: in the absence of retardation effects and of inhomogeneity it is given by Eq. (10.5) and in the general case by Eqs (15.5)-(15.8). The second term is the part of the collision integral which is determined by the long-range correlations:

$$\tilde{I} = n \int dx_2 \frac{\partial \Phi_{12}}{\partial \boldsymbol{r}_1} \cdot \frac{\partial}{\partial \boldsymbol{p}_1} \tilde{g}_2(x_1, x_2, t) . \tag{21.2}$$

In the same approximation one can use Eq. (18.25) for the correlation function $\tilde{g}_2$ instead of Eq. (18.13) for the function $\tilde{f}_2$.

We already noted that in Eq. (18.25) the last term describes the contributions of triple and higher-order collisions. These can be divided in turn into 'polarization' terms (see sect. 3) which are defined by two-body correlation functions $\tilde{g}_2(2,3)$, $\tilde{g}_2(1,3)$, and terms containing three-body correlations.

Here we shall calculate the fluctuations in the gas in the binary correlation approximation; hence we shall set $\tilde{g}_3 = 0$ in Eq. (18.2). Besides, because of the smallness of the long-range correlation functions, one may neglect in Eq. (18.25) the term $\hat{\theta}_{12}\, \tilde{g}_2$ as compared to $\hat{\theta}_{12} f_1 f_1$ on the right-hand side. As a result, we obtain the following equation for the function $\tilde{g}_2$.

---

† **A thorough treatment of long-range fluctuations, from a very different point of view is given in the recent book by Nicolis and Prigogine [72]. (Translator.)**

$$\left(\hat{L}_{x_1x_2} + \delta\hat{I}_{x_1} + \delta\hat{I}_{x_2}\right)\tilde{g}_2(1,2) = \hat{\theta}_{12}\, f_1(1)\, f_1(2)$$
$$- \frac{1}{V}\int dx_3(\hat{\theta}_{13}+\hat{\theta}_{23})\; f_1(1)\, f_1(2)\, f_1(3) + \overline{\theta_{12} f_1(\boldsymbol{r}_1, \boldsymbol{P}_1, t)\; f_1(\boldsymbol{r}_1, \boldsymbol{P}_2, t)}$$
$$- \frac{1}{N}\left[I_B(x_1,t)\; f_1(2) + I_B(x_2,t)\; f_1(1)\right]$$
$$+ n\int dx_3[\hat{\theta}_{13}\tilde{g}_2(2,3)\, f_1(1) + \hat{\theta}_{23}\tilde{g}_2(1,3)\, f_1(2)] \quad . \tag{21.3}$$

This equation is analogous in structure to Eq. (3.5) for the pair correlation function $g_2(x_1,x_2,t)$ obtained in the polarization approximation from the BBGKY hierarchy.

The main difference with this equation lies in the presence, on the left-hand side of Eq. (21.3), of the additional terms $\delta\hat{I}_{x_1}\tilde{g}_2$, $\delta\hat{I}_{x_2}\tilde{g}_2$. They describe the dissipative and non-dissipative processes due to the binary collisions.

The polarization terms, i.e. the last terms on the right hand side of Eqs (21.3),(3.5) have the same form.

The first four terms on the right hand side of Eq. (21.3) are defined in terms of the one-particle distribution functions f_1. They can be considered as sources in the equation for $\tilde{g}_2$. The first term of the source corresponds to the source of Eq. (3.5). The second term compensates the first when integrated over x_1 or x_2. The third term in the source has the form

$$\overline{\hat{\theta}_{12} f_1(\boldsymbol{r}_1, \boldsymbol{P}_1(-\infty), t)\;\; f_1(\boldsymbol{r}_1, \boldsymbol{P}_2(-\infty), t)} \quad . \tag{21.4}$$

There appears here the symbol of smoothing, i.e. averaging over a physically infinitesimal volume $l_\phi^3 \sim (r_0/\varepsilon^{\frac{1}{2}})^3 \gg r_0^3$.

In the state of local equilibrium, when $f_1 = f_1^0$, the Maxwell distribution, the expression (21.4) goes to zero because the integrand is an odd function of $\boldsymbol{r}_{12}$ and the averaging integral vanishes.

The fourth term in the source supplements the third. Together with the latter it gives zero by integration over x_1 or x_2, just like the first two terms. Hence, the equation for $V^{-1}\int dx_2\,\tilde{g}_2$ is homogeneous and has the solution

$$\frac{1}{V}\int dx_2\,\tilde{g}_2 = 0 \quad . \tag{21.5}$$

The necessity of this condition follows from the definition (18.14) of the correlation function $\tilde{g}_2$.

22. EQUATION FOR THE SMOOTHED PHASE DENSITY. THE METHOD OF MOMENTS

In sections 4-6 we used as a basis for the construction of the non-equilibrium statistical theory of gases the equation (4.6) for the phase density in the six-dimensional space of positions and momenta. Averaging that equation we obtained

a hierarchy for the moments of the phase density, which is equivalent to the BBGKY hierarchy.

In particular, in section 6 we obtained Eq. (6.13) for the second one-time moment of the phase-density fluctuation $\langle \delta N\, \delta N \rangle_{x,x',t}$ which is equivalent to Eq. (3.5) for the two-particle correlation function g_2 in the polarization approximation. It was also shown that, for the description of the processes in this approximation, we can also take as a starting point Eq. (6.32) for the fluctuations of the phase density δN. This equation contains a source $\delta N^{\mathrm{source}}$. The correlations of the source fluctuations is defined by Eq. (6.33).

Similar considerations can be applied to the study of the long-range fluctuations in gases. To this purpose we take, as a starting point, the equation for the smoothed phase density $\tilde{N}(x,t)$ instead of Eq. (4.6). From the former we obtain, by averaging, a hierarchy for the moments, equivalent to the hierarchy for the smoothed functions $f_1, \tilde{f}_2, \tilde{f}_3, \ldots$.

We rewrite the equation for $\tilde{N}(x,t)$ in a form analogous to Eq. (4.7)

$$\left(\frac{\partial}{\partial t} + \boldsymbol{v} \cdot \frac{\partial}{\partial \boldsymbol{r}} + \boldsymbol{F}_0 \cdot \frac{\partial}{\partial \boldsymbol{p}}\right) \tilde{N}(x,t) = \int dx'\, \hat{\theta}_{xx'} \left[\tilde{N}(x,t)\, \tilde{N}(x',t) + \tilde{N}(\boldsymbol{r}, \boldsymbol{P}(-\infty), t)\, \tilde{N}(\boldsymbol{r}, \boldsymbol{P}'(-\infty), t)\right] \tag{22.1}$$

where the operator $\hat{\theta}_{xx'}$ is analogous to the operator $\hat{\theta}_{ij}$, (1.6)

$$\hat{\theta}_{xx'} = \frac{\partial \Phi(|\boldsymbol{r}-\boldsymbol{r}'|)}{\partial \boldsymbol{r}} \cdot \frac{\partial}{\partial \boldsymbol{p}} + \frac{\partial \Phi(|\boldsymbol{r}-\boldsymbol{r}'|)}{\partial \boldsymbol{r}'} \cdot \frac{\partial}{\partial \boldsymbol{p}'} . \tag{22.2}$$

The last term in (22.1), describing the contribution of the pair collisions, is written in the ideal-gas approximation. If the retardation and inhomogeneity effects are taken into account to first order in $\tau_0/\tau_{\mathrm{rel}}$, r_0/l , this term must be written as follows:

$$\int dx'\, \hat{\theta}_{xx'} \left\{\left(1 - \frac{\boldsymbol{r}-\boldsymbol{r}'}{2} \cdot \frac{\partial}{\partial \boldsymbol{r}}\right)\left[\tilde{N}(\boldsymbol{r}, \boldsymbol{P}(-\infty), t)\tilde{N}(\boldsymbol{r}, \boldsymbol{P}'(-\infty), t) - \tilde{N}(\boldsymbol{r}, \boldsymbol{p}, t)\, \tilde{N}(\boldsymbol{r}, \boldsymbol{p}', t)\right] - \left(\frac{\partial}{\partial t} + \frac{\boldsymbol{v}+\boldsymbol{v}'}{2} \cdot \frac{\partial}{\partial \boldsymbol{r}} + \boldsymbol{F}_0 \cdot \frac{\partial}{\partial \boldsymbol{p}} + \boldsymbol{F}_0 \cdot \frac{\partial}{\partial \boldsymbol{p}'}\right) \cdot \int_0^\infty d\tau\, \tau \frac{d}{d\tau} \tilde{N}(\boldsymbol{r}, \boldsymbol{P}(-\tau), t)\, \tilde{N}(\boldsymbol{r}, \boldsymbol{P}'(-\tau), t)\right\} . \tag{22.3}$$

This expression is written in analogy with (15.6) – (15.8). Equation (22.1) for the phase density, in the special case $\boldsymbol{F}_0 = 0$ and without the first term on the right-hand side was first used by Kadomtsev for the calculation of fluctuations in an ideal gas [25] †.

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† **In a recent preprint of N.N. Bogolyubov [66] the equation for the microscopic phase density corresponding to the Boltzmann-Enskog equation (sect. 15) is considered.**

The equation for the smoothed phase density can be obtained from the exact equation (4.6) for $N(x,t)$ by the same method by which Eq. (18.10) for the smoothed distribution $\tilde{f}_N$ was obtained from the Liouville equation. We now proceed to the averaging of Eq. (22.1). Hereafter, wherever no confusion can arise, the tilde ~ will be omitted over the function $N(x,t)$, i.e. $\tilde{N}(x,t) \to N(x,t)$.

The relations (5.1), (5.2), (5.6) hold true for the smoothed functions as well:

$$\langle \tilde{N}(x,t) \rangle = n f_1(x,t) \tag{22.4}$$

$$\langle \tilde{N}(x,t)\tilde{N}(x',t)\rangle = \frac{N(N-1)}{V^2} \tilde{f}_2(x,x',t) + \frac{N}{V} \delta(x-x') f_1(x,t) \tag{22.5}$$

$$\begin{aligned}\langle \delta N \delta N\rangle_{x,x',t} &= \frac{N(N-1)}{V^2} \tilde{g}_2(x,x',t) \\ &+ \frac{N}{V}\left[\delta(x-x') f_1(x,t) - \frac{1}{V} f_1(x,t) f_1(x',t)\right] .\end{aligned} \tag{22.6}$$

Here

$$\delta N(x,t) = \tilde{N}(x,t) - \langle \tilde{N}(x,t)\rangle \tag{22.7}$$

is the fluctuation of the smoothed phase density.

The equation for the average phase density $\langle \tilde{N} \rangle$ [see Eqs (22.4), (22.5)] corresponds to Eq. (18.11): the first of the hierarchy for the smoothed functions $f_1, \tilde{f}_2, \ldots$. If the contribution of $\tilde{g}_2$ for distances of order r_0 is neglected, as was done in the passage from (18.11) to (18.16), we obtain the following equation

$$\left(\frac{\partial}{\partial t} + \boldsymbol{v} \cdot \frac{\partial}{\partial \boldsymbol{r}} + \boldsymbol{F} \cdot \frac{\partial}{\partial \boldsymbol{p}}\right) f_1(x,t) = I_B(x,t) + \tilde{I}(x,t) . \tag{22.8}$$

This equation differs from (18.16) only in the fact that the contribution of the long-range correlations to the collision integrals is defined by the following equation [instead of (18.17)]:

$$\tilde{I}(x,t) = \frac{1}{n}\int dx' \, \hat{\theta}_{xx'} \langle \delta N \; \delta N\rangle_{x,x',t} \equiv \frac{1}{n} \frac{\partial}{\partial \boldsymbol{p}} \cdot \langle \delta \boldsymbol{F} \, \delta N\rangle_{\boldsymbol{r},x,t} . \tag{22.9}$$

As in (18.16), I_B is the Boltzmann collision integral, defined by (10.5) in the absence of retardation and inhomogeneity effects, and by (15.5) – (15.8) in the general case.

From Eq. (22.8), defining the relation between $\langle \delta N \delta N\rangle_{x,x',t}$ and $\tilde{g}_2$ follows that there are two additional terms in (22.9) when compared to (18.17). The second term on the right-hand side of Eq. (22.6) takes account of the self-interaction; it vanishes upon integration over r' in the integral $\tilde{I}$. The second additional term (originating from the last term in (22.6)) takes account of the fact that in the expression for the average force (2.10) there occurs a factor N/V instead of $(N-1)/V$: this difference is compensated by this term.

Expression (22.6), just like the corresponding one (5.6), satisfies the condition:

$$\frac{1}{V}\int dx' \left\langle \delta N\ \delta N \right\rangle_{x,x',t} = \frac{1}{V}\int dx \left\langle \delta N\ \delta N \right\rangle_{x,x',t} = 0 \qquad (22.10)$$

which follows from the conservation of the number of particles.

From Eq, (22.11) we now derive the second equation of the hierarchy for the moments of the smoothed phase density $\tilde{N}$. Omitting the tilde, we obtain:

$$L^0_{xx'}\left\langle NN\right\rangle_{x,x',t} = \int dx'' \left\{ \theta_{xx''}\left[\left\langle NNN\right\rangle_{x,x',x'',t} + \left\langle NNN\right\rangle_{\boldsymbol{P},x',\boldsymbol{P}'',t}\right]\right.$$
$$\left. + \hat{\theta}_{x'x''}\left[\left\langle NNN\right\rangle_{x,x',x'',t} + \left\langle NNN\right\rangle_{x,\boldsymbol{P}',\boldsymbol{P}'',t}\right]\right\} \qquad (22.11)$$

This equation corresponds to Eq. (18.1) for the smoothed two-particle function $\tilde{f}_2$.

Let us show how we can make the transition from Eq. (22.11) to Eq. (18.13). This transition will allow us to establish the rule defining the action of the collision operator on terms containing singular functions. Such functions appear in the expression relating the moments of the phase density to the distribution functions, e.g. in Eq. (22.5).

We write the corresponding expression for the third moment

$$\left\langle NNN\right\rangle_{x,x',x'',t} = \frac{N(N-1)(N-2)}{V^3}\,\tilde{f}_3\,(x,x',x'',t)$$
$$+ \frac{N(N-1)}{V^2}\left[\delta(x-x')\,\tilde{f}_2(x',x'',t) + \delta(x-x'')\tilde{f}_2(x'',x,t)\right.$$
$$\left. + \delta(x''-x)\,\tilde{f}_2(x,x',t)\right] + \frac{N}{V}\,\delta(x-x')\ \delta(x'-x'')f_2\,(x'',t). \qquad (22.12)$$

We substitute this expression into the first terms enclosed in square brackets on the right-hand side of (22.11). We neglect the terms describing self-interaction, we integrate the terms containing $\delta(x-x'')$, $\delta(x'-x'')$ over x'' and note that under the integrals in (22.11)

$$\left(\hat{\theta}_{xx''} + \hat{\theta}_{x'x''}\right)\delta(x-x')\ldots = \delta(x-x')\left(\hat{\theta}_{xx''} + \hat{\theta}_{x'x''}\right)\ldots \qquad (22.13)$$

We thus find

$$\int dx''\left[\hat{\theta}_{xx''}\left\langle NNN\right\rangle_{x,x',x'',t} + \hat{\theta}_{x'x''}\left\langle NNN\right\rangle_{x,x',x'',t}\right]$$
$$= \frac{N(N-1)(N-2)}{V^3}\int dx''\left(\hat{\theta}_{xx''} + \hat{\theta}_{x'x''}\right)\tilde{f}_3(x,x',x'',t)$$
$$+ \frac{N(N-1)}{V^2}\left\{\frac{1}{2}\,\delta(x-x')\int dx''\left[\hat{\theta}_{xx''}\,\tilde{f}_2(x,x'',t) + \hat{\theta}_{x'x''}\,\tilde{f}_2(x',x'',t)\right]\right.$$
$$\left. + \hat{\theta}_{xx'}\,\tilde{f}_2(x,x',t)\right\}. \qquad (22.14)$$

The factor $\frac{1}{2}$ before $\delta(x-x')$ appears from a symmetrization (with respect to x,x') of the expression enclosed in square brackets on the right-hand side of (22.14).

Consider now the remaining two terms in (22.11). They differ from the previous ones only in the following change of arguments:

$$x,x'' \to \boldsymbol{r},\boldsymbol{P},\boldsymbol{r},\boldsymbol{P}'' , \qquad x',x'' \to \boldsymbol{r}',\boldsymbol{P}',\boldsymbol{r}',\boldsymbol{P}'' . \tag{22.15}$$

Clearly, this change effects only the arguments of the distribution functions of the interacting particles. Thus the substitution (22.15) into Eq. (22.11), combined with (22.12), does not change the arguments of the delta functions, and we obtain:

$$\begin{aligned}
&\int dx''\left[\hat{\theta}_{xx''}\langle NNN\rangle_{\boldsymbol{P},x',\boldsymbol{P}'',t}+\hat{\theta}_{x'x''}\langle NNN\rangle_{x\ \boldsymbol{P}',\boldsymbol{P}'',t}\right]\\
&\quad=\frac{N(N-1)(N-2)}{V^3}\int dx''\left[\hat{\theta}_{xx''}\tilde{f}_3(\boldsymbol{P},x',\boldsymbol{P}'',t)+\theta_{x'x''}\tilde{f}_3(x,\boldsymbol{P}',\boldsymbol{P}'',t)\right]\\
&\quad+\frac{N(N-1)}{V^2}\left\{\tfrac{1}{2}\,\delta(x-x')\int dx''\left[\hat{\theta}_{xx''}f_2(\boldsymbol{P},\boldsymbol{P}'',t)+\hat{\theta}_{x'x''}\tilde{f}_2(\boldsymbol{P}',\boldsymbol{P}'',t)\right]\right.\\
&\quad\left.+\,\hat{\theta}_{xx'}\ \tilde{f}_2(\boldsymbol{P},\boldsymbol{P}',t)\right\} .
\end{aligned}\tag{22.16}$$

We now transform the left-hand side of Eq. (22.11). We substitute into it Eq. (22.5) and consider the contribution of the second term of (22.5). Performing a symmetrization and using the first equation of the hierarchy for the smoothed functions, i.e. Eq. (22.8), we obtain

$$\begin{aligned}
&\hat{L}^0_{xx'}\ \frac{n}{2}\ \delta(x-x')\left[f_1(x,t)+f_1(x',t)\right]\\
&\quad=\frac{n}{2}\ \delta(x-x')\left[\hat{L}^0_x\ f_1(x,t)+\hat{L}^0_{x'}\,f_1(x',t)\right]\\
&\quad=\frac{1}{2}\ \frac{N(N-1)}{V^2}\ \delta(x-x')\int dx''\left\{\hat{\theta}_{xx''}\left[\tilde{f}_2(x,x'',t)+\tilde{f}_2(\boldsymbol{P},\boldsymbol{P}'',t)\right]\right.\\
&\quad\left.+\,\hat{\theta}_{x'x''}\left[\tilde{f}_2(x',x'',t)+\tilde{f}_2(\boldsymbol{P}',\boldsymbol{P}'',t)\right]\right\}
\end{aligned}\tag{22.17}$$

Using (22.14), (22.16), (22.17), we find from Eq. (22.11) an equation for $\tilde{f}_2$ identical with Eq. (18.13), i.e. the second equation of the hierarchy for the smoothed functions $f_1, \tilde{f}_2 \ldots$. The same procedure applies to the remaining equations.

Thus, the hierarchy of equations for the momenta of the phase density $\tilde{N}(x,t)$, obeying Eq. (22.1), is equivalent to the hierarchy for the smoothed distribution functions which follows from the Liouville equation for $\tilde{f}_N$. We note again that in the proof of the equivalence, the substitution (22.15) appeared in the moment equations only in the arguments of the distribution functions.

From the proof of the equivalence it follows that, in the polarization approximation, we obtain from the moment hierarchy an equation identical to (21.3). We rewrite it in a form better adapted to the moment method. Using Eq. (22.6), we can group together on the right-hand side of (21.3) the last term and the first two terms. The equation then becomes $(x_1, x_2, x_3 \to x, x', x'')$

$$
\begin{aligned}
&\left(\hat{L}_{xx'} + \delta\hat{I}_x + \delta\hat{I}_{x'}\right) \tilde{g}_2(x,x',t) \\
&\quad = \frac{1}{n}\int dx'' \Big[\hat{\theta}_{xx''} \left\langle \delta N \ \delta N\right\rangle_{x',x'',t} \ f_1(x,t) \\
&\quad + \hat{\theta}_{x'x''}\left\langle \delta N\ \delta N\right\rangle_{x,x'',t} \ f_1(x',t)\Big] + \hat{\theta}_{xx'} \ \overbrace{f_1(\boldsymbol{P},t)\, f_1(\boldsymbol{P}',t)} \\
&\quad - \frac{1}{N}\Big[I_B(x,t)\ f_1(x',t) + I_B(x',t)\ f_1(x,t)\Big] . \qquad (22.18)
\end{aligned}
$$

We continue the transformation of this equation. We multiply Eq. (22.18) by n^2, add to both sides the expression

$$
\left(\hat{L}_{xx'} + \delta\hat{I}_x + \delta\hat{I}_{x'}\right) n\left[\delta(x-x')f_1 - \frac{1}{V} f_1(x,t)\ f_1(x',t)\right] \qquad (22.19)
$$

and use Eq. (22.6) together with the relation between $\delta\boldsymbol{F}$ and δN:

$$
\delta\boldsymbol{F}(\boldsymbol{r},t) = -\frac{\partial}{\partial \boldsymbol{r}} \int dx' \ \Phi_{\boldsymbol{r}\boldsymbol{r}'} \ \delta N(x',t) \ . \qquad (22.20)
$$

As a result we obtain the following equation for the second moment $\left\langle \delta N\ \delta N\right\rangle_{x,x',t}$:

$$
\begin{aligned}
&\left(\hat{L}_{xx'} + \delta\hat{I}_x + \delta\hat{I}_{x'}\right)\left\langle \delta N\ \delta N\right\rangle_{x,x',t} + \\
&\quad + \left\langle \delta\boldsymbol{F}\ \delta N\right\rangle_{\boldsymbol{r},x',t} \cdot \frac{\partial n f_1(x,t)}{\partial \boldsymbol{p}} + \left\langle \delta N\ \delta\boldsymbol{F}\right\rangle_{x,\boldsymbol{r}',t} \cdot \frac{\partial n f_1(x',t)}{\partial \boldsymbol{p}'} \\
&\quad = A(x,x',t) , \qquad (22.21)
\end{aligned}
$$

where

$$
\begin{aligned}
\mathrm{A}(x,x',t) &= \left(\hat{L}_{xx'} + \delta\hat{I}_x + \delta\hat{I}_{x'}\right) n\left[\delta(x-x')f_1 - \frac{1}{V} f_1(x,t)\ f_1(x',t)\right] \\
&\quad + n^2\,\hat{\theta}_{xx'}\ f_1(\boldsymbol{P},t)\, f_1(\boldsymbol{P}',t) - \frac{n}{V}\Big[I_B(x,t)\, f_1(x',t) + I_B(x',t)\, f_1(x,t)\Big] . \qquad (22.22)
\end{aligned}
$$

The function A is defined by one-particle distribution functions, and thus plays the role of a source in the equation for the second moments. We have in this respect an analogy with Eq. (6.13). Let us transform (22.22). Assuming, as before, that under the action of $\delta\hat{I}_x$, $\delta\hat{I}_{x'}$, the change (22.15) effects only the arguments of the distribution functions, and using Eq. (22.8) for f_1 we find

$$
A(x,x',t) \equiv \tilde{A} = n\left\{\delta(x-x')\,\tilde{I}(x,t) - \frac{1}{V}\left[\tilde{I}(x,t)\, f_1(x',t) + \tilde{I}(x',t)\, f_1(x,t)\right]\right\} . \qquad (22.23)
$$

It follows that A obeys the conditions

$$\int dx\, A(x,x',t) = \int dx'\, A(x,x',t) = 0\,. \tag{22.24}$$

We note that the right-hand side of Eq. (6.13) can also be brought into the form (22.23): using Eq. (6.2) we obtain

$$\hat{L}_{xx'} \langle \delta N\, \delta N\rangle^{\text{source}}_{x,x',t} = n\Big\{\delta(x-x')I(x,t) - \frac{1}{V}\Big[I(x,t)\,f_1(x',t) + I(x',t)f_1(x,t)\Big]\Big\}\,. \tag{22.25}$$

We see that the source (22.23) is expressed in terms of $\tilde{I}$, the part of the collision integral related to the long-range fluctuations, as is to be expected. Indeed, when $\tilde{I}=0$, i.e. when the long-range fluctuations are not taken into account, the Boltzmann equation, in the approximation considered here (see sect. 10), gives an exhaustive description of the kinetic processes in gases in the binary collision regime.

Equations (22.8) and (22.18) form a closed set of equations for the functions $f_1, \tilde{g}_2$. They can be taken as a basis for the description of non-equilibrium processes in gases, taking into account the long-range fluctuations. Alternatively, we may use Eqs (22.8) and (22.21) for the functions f_1, $\langle \delta N\, \delta N\rangle_{x,x',t}$.

From the set of equations for $f_1, \tilde{g}_2$ (or f_1, $\langle\delta N\ \delta N\rangle$) it is impossible to obtain a closed set of equations for f_1, i.e. a kinetic equation, without further simplifying assumptions. The transition to a kinetic equation is possible if the function $f_1(x,t)$ changes slowly over distances of order l and times of order τ_{rel}. In that case we may, as in the derivation of the Boltzmann equation, use the condition of weakening of the initial correlations, express $\tilde{g}_2$ in terms of f_1, and thus obtain an explicit expression for the collision integral in (22.8). This will be done in Chpater 11, in the case of plasmas.

We may drop still another limitation (condition 4 of sect. 16) which was necessary for the derivation of the Boltzmann equation. It was assumed that the collision process of any atom within a physically infinitesimal volume element can be considered as a continuous process. Actually, the collision process is discrete ('shot effect').

The discreteness of the collision process leads to an additional contribution to the source on the right-hand side of Eq. (22.21)

$$A(x,x',t) = \tilde{A}(x,x',t) + A^{\mathbf{B}}(x,x',t) \tag{22.26}$$

$\tilde{A}$ is expressed in terms of the collision integral $\tilde{I}$ (see (22.23)). The second term differs from zero even if $\tilde{I}=0$. As will be seen, it is defined by the same processes as in the collision integral I_B: we therefore use the superscript **B** for this term.

The shot effect, i.e. the discreteness of the collision process, is taken into account as follows. In the function (22.22), whenever the operators act explicitly, the substitution (22.15) must be made in all arguments involved in the collision operator†. We thus find, instead of (22.23), Eq. (22.26) together with

$$A^{\mathbf{B}}(x,x',t) = n^2 \int dx'' \Big\{ \hat{\theta}_{xx''} \Big[\delta(\boldsymbol{r}'-\boldsymbol{r}) \Big(\delta(\boldsymbol{p}'-\boldsymbol{p}) - \delta(\boldsymbol{p}'-\boldsymbol{P}) \Big) + \delta(\boldsymbol{r}'-\boldsymbol{r}'') \Big(\delta(\boldsymbol{p}'-\boldsymbol{p}'') - \delta(\boldsymbol{p}'-\boldsymbol{P}'') \Big) \Big] f_1(\boldsymbol{P},t)\, f_1(\boldsymbol{P}'',t) + (\boldsymbol{r},\boldsymbol{p} \rightleftharpoons \boldsymbol{r}',\boldsymbol{p}') \Big\} . \tag{22.27}$$

Clearly, $A^{\mathbf{B}} = 0$ if the substitution (22.15) only affects the arguments of the distribution functions. The function $A^{\mathbf{B}}$ satisfies the conditions (22.24).

In the equilibrium state the function $\tilde{A} = 0$, because $I = 0$. But the function $A^{\mathbf{B}} \neq 0$ even in equilibrium, because the fluctuations due to the discrete collision processes exist also in equilibrium. In this case the expression (22.27) simplifies [the terms containing the functions $\delta(\boldsymbol{p}'-\boldsymbol{p})$, $\delta(\boldsymbol{p}'-\boldsymbol{p}'')$, $\delta(\boldsymbol{p}-\boldsymbol{p}'')$ vanish]:

$$A^{\mathbf{B}} = - n^2 \int dx'' \Big\{ \hat{\theta}_{xx''} \Big[\delta(\boldsymbol{r}-\boldsymbol{r}')\, \delta(\boldsymbol{p}'-\boldsymbol{P}) + \delta(\boldsymbol{r}'-\boldsymbol{r}'')\, \delta(\boldsymbol{p}'-\boldsymbol{P}'') \Big] f_1(\boldsymbol{P})\, f_1(\boldsymbol{P}'') + (\boldsymbol{r},\boldsymbol{p} \rightleftharpoons \boldsymbol{r}',\boldsymbol{p}') \Big\} . \tag{22.28}$$

A different form of $A^{\mathbf{B}}$ will be convenient for further use. It follows from (22.27) upon integration over $\boldsymbol{r}'', \boldsymbol{p}''$ in the terms containing the functions $\delta(\boldsymbol{r}'-\boldsymbol{r}'')\,\delta(\boldsymbol{p}'-\boldsymbol{p}'')$, $\delta(\boldsymbol{r}-\boldsymbol{r}'')\,\delta(\boldsymbol{p}-\boldsymbol{p}'')$, and upon putting the functions $\delta(\boldsymbol{r}'-\boldsymbol{r})\,\delta(\boldsymbol{p}'-\boldsymbol{p})$ and $\delta(\boldsymbol{r}-\boldsymbol{r}')\,\delta(\boldsymbol{p}-\boldsymbol{p}')$ under the $\hat{\theta}_{xx''}$, $\hat{\theta}_{x'x''}$ operators; we obtain

$$A^{\mathbf{B}}(x,x',t) = n\,\delta(x-x')\, I_{\mathbf{B}}(x,t) + n\, I_{xx'} - n^2 \int dx'' \Big\{ \hat{\theta}_{xx''} \Big[\delta(\boldsymbol{r}'-\boldsymbol{r})\, \delta(\boldsymbol{p}'-\boldsymbol{P}) + \delta(\boldsymbol{r}'-\boldsymbol{r}'')\, \delta(\boldsymbol{p}'-\boldsymbol{P}'') \Big] f_1(\boldsymbol{P},t)\, f_1(\boldsymbol{P}'',t) + (\boldsymbol{r},\boldsymbol{p} \rightleftharpoons \boldsymbol{r}',\boldsymbol{p}') \Big\} \tag{22.27$_2$}$$

where $I_{\mathbf{B}}(x,t)$ is Boltzmann's collision integral, and

$$I_{xx'} = \overbrace{n\, \hat{\theta}_{xx'}\, f_1(\boldsymbol{P},t)\, f_1(\boldsymbol{P}',t)} . \tag{22.29}$$

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† **This amounts to admitting a variation of the function f_1 over the range $\sim l_\phi$, i.e., over the 'width' of the δ-functions.**

It then follows that

$$\int dx' \, I_{xx'} = I_{\mathbf{B}}(x,t) \,, \qquad \int dx \, I_{xx'} = I_{\mathbf{B}}(x',t) \,.$$

In equilibrium, the first two terms of $(22.27)_2$ vanish, and we are left again with (22.28).

Finally, still another form of the function $A^{\mathbf{B}}(x,x',t)$ is useful: it helps in understanding the physical meaning of the source of the fluctuations:

$$A^{\mathbf{B}}(x,x',t) = [(\delta\hat{I}_x + \delta\hat{I}_{x'}) - (\delta\hat{I}_x + \delta\hat{I}_{x'})_0] \, n \, \delta(x-x') \, f(x,t). \qquad (22.27)_3$$

The subscript 0 means that the corresponding operators act only on the distribution function $f(x,t)$, and not on the δ-function.

In equilibrium, the second term in $(22.7)_3$ vanishes and the function $A^{\mathbf{B}}$ reduces to

$$A^{\mathbf{B}}(x,x',t) = (\delta\hat{I}_x + \delta\hat{I}_{x'}) \, n \, \delta(x-x') \, f(x,t) \,. \qquad (22.28)_2$$

This result was first obtained by Kadomtsev [25] .

The formulae obtained here can be studied also for model collision integrals. For instance, in the ν-approximation, $A^{\mathbf{B}}$ vanishes, but it is different from zero for the Bathnagar-Gross-Krook collision integral.

For the Fokker-Planck collision operator, when

$$\delta\hat{I} = -\left(D\frac{\partial^2}{\partial p^2} + \frac{\partial}{\partial p}\,\gamma p\right)$$

$$D = \gamma m k_B T$$

We find from $(22.27)_3$:

$$A^{\mathbf{F-P}} = 2\,D\,\frac{\partial^2}{\partial \boldsymbol{p}\,\partial \boldsymbol{p}'}\,\delta(\boldsymbol{r}-\boldsymbol{r}')\,\delta(\boldsymbol{p}-\boldsymbol{p}')\,f(\boldsymbol{r},\boldsymbol{p},t) \,.$$

Thus, in equilibrium $\tilde{A} = 0$, but $A^{\mathbf{B}} \neq 0$. Such a disparity is quite natural. It is due to the fact that in the hierarchy for the smoothed functions, or in the corresponding moment hierarchy, the dissipation is due to the short-range fluctuations which define the Boltzmann collision integral. The present equations define the statistical properties of the long-range fluctuations. The spectral density of these fluctuations will be studied in the next section. It will be seen that the collision integral $\tilde{I}$ can be expressed through the latter, if the characteristic scales T, L for the functions f_1 satisfy $T \gg \tau_{\mathbf{rel}}$, $L \gg l$. This occurs, for instance, in the passage to the hydrodynamic description. Naturally, in the hydrodynamic equations, the dissipation due to both the short-range and the long-range fluctuations is taken into account. In other words, the dissipative contributions will be expressed not only in terms of the Boltzmann collision integral $I_{\mathbf{B}}$, but also in terms of $\tilde{I}$.

For the calculation of the space-time density of the long-range fluctuations we need an equation for the two-time moments or their corresponding correlation functions. The equation for $\tilde{g}_2(x,t,x',t')$ in the polarization approximation can be obtained by analogy to Eq. (22.18) for the one-time function $\tilde{g}_2(x,x',t)$. It has the form

$$\left(L_x + \delta\hat{I}_x\right) \tilde{g}_2(x,t,x',t') = \frac{1}{n}\int dx''\, \hat{\theta}_{xx'} \left\langle \delta N\, \delta N \right\rangle_{x'',t,x',t'} f_1(x,t)$$
$$+ \int dx''\, \theta_{xx''} f_1(\boldsymbol{r},\boldsymbol{P},t)\, F_1(x',t',\boldsymbol{r},\boldsymbol{P}'',t)$$
$$-\frac{1}{N} I_B(x,t)\, f_1(x',t') \;, \tag{22.30}$$

where the function $F_1(x,t,x',t')$ was defined in Eq. (5.11).

Equation (22.18) for the one-time correlation $\tilde{g}_2(x,x',t)$ can be obtained from (22.30). To this purpose, one must write the second equation (22.30) for $\tilde{g}_2(x',t,x,t')$, combine the two equations and take the limit $t'\to t$. This limiting process in the second term of the right-hand side of (22.30) is performed as follows. We introduce the retardation and inhomogeneity, and at the end take $\tau = 0$, $\boldsymbol{r}_3 = \boldsymbol{r}$:

$$\lim_{t'\to t} \int dx_3\, \hat{\theta}_{13} f_1(\boldsymbol{r}_1,\boldsymbol{P}_1,t)\, F_1(x_2,t'-\tau,\boldsymbol{r}_3,\boldsymbol{P}_3,t-\tau)$$
$$= \int dx_3\, \hat{\theta}_{13} f_1(\boldsymbol{r}_1,\boldsymbol{P}_1,t)\, \delta(\boldsymbol{r}_2-\boldsymbol{r}_3)\, \delta(\boldsymbol{P}_2-\boldsymbol{P}_3)\, f_1(\boldsymbol{r}_3,\boldsymbol{P}_3,t-\tau)\Big|_{\boldsymbol{r}_2=\boldsymbol{r}_1,\tau=0}$$
$$= \hat{\theta}_{12}\, f_1(\boldsymbol{r}_1,\boldsymbol{P}_1,t)\, f_1(\boldsymbol{r}_1,\boldsymbol{P}_2,t) \,. \tag{22.31}$$

We rewrite the two-time equation by using the following equality:

$$\left\langle \delta N\, \delta N \right\rangle_{x,t,x',t'} = \frac{N(N-1)}{V^2}\, \tilde{g}_2(x,t,x',t')$$
$$+ n\left[F_1(x,t,x',t') - \frac{1}{V}\, f_1(x,t)\, f_1(x',t')\right] \tag{22.32}$$

which, for $t'=t$ goes over into (22.6). We multiply (22.30) by $N(N-1)/V^2$ and add to both sides the expression

$$\left(\hat{L}_x + \delta\hat{I}_x\right) n\left[F_1(x,t,x',t') - \frac{1}{V}\, f_1(x,t)\, f_1(x',t')\right] . \tag{22.33}$$

We thus find an equation for the two-time moment:

$$\left(\hat{L}_x + \delta\hat{I}_x\right) \left\langle \delta N\ \delta N\right\rangle_{x,t,x',t'} + \left\langle \delta \boldsymbol{F}\ \delta N \right\rangle_{\boldsymbol{r},t,x',t'} \cdot \frac{\partial n f_1(x,t)}{\partial \boldsymbol{p}}$$
$$= A(x,t,x',t') \tag{22.34}$$

where the two-time source is given by

$$A(x,t,x',t') = \left(\hat{L}_x + \delta\hat{I}_x\right) n \left[F_1(x,t,x',t') - \frac{1}{V} f_1(x,t)\, f_1(x',t')\right]$$
$$+ n^2 \int dx''\, \hat{\theta}_{xx''}\, f_1(\boldsymbol{r},\boldsymbol{P},t)\, F_1(x',t',\boldsymbol{r},\boldsymbol{P}'',t)$$
$$- \frac{n}{V}\, I_B(x,t)\, f_1(x',t') \, . \tag{22.35}$$

Equation (22.34) corresponds to Eq. (22.21). Let us transform expression (22.35) for the source $A(x,t,x',t')$: we use the equation for the function $F_1(x,t,x',t')$:

$$\hat{L}_x F_1 = \frac{N-1}{V} \int dx''\, \hat{\theta}_{xx''} \Big[\tilde{g}_2(x,t,x',t';x'',t)$$
$$+ F_1(\boldsymbol{r},\boldsymbol{P},t,x',t') f_1(\boldsymbol{r},\boldsymbol{P}'',t)\Big] \, . \tag{22.36}$$

This equation follows directly from the Liouville equation for the two-time smoothed distribution function (sect. 18) or from the corresponding equation for the phase density $\tilde{N}(x,t,x',t')$. The latter differs from (22.1) only by the change $\tilde{N}(x,t) \to \tilde{N}(x,t,x',t')$.

Let us write explicitly the expression for $\delta\hat{I}_x F_1$ in (22.35)

$$\delta\hat{I}_x F_1(x,t,x',t') = -\frac{N-1}{V} \int dx''\, \hat{\theta}_{xx''} \Big[F_1(\boldsymbol{r},\boldsymbol{P},t,x',t')\, f_1(\boldsymbol{r},\boldsymbol{P}'',t)$$
$$+ f_1(\boldsymbol{r},\boldsymbol{P},t)\, F_1(\boldsymbol{r},\boldsymbol{P}'',t,x',t')\Big] \, . \tag{22.37}$$

Using this expression and Eq. (22.36) as well as the kinetic equation for f_1, we obtain the following expression for A:

$$A(x,t,x',t') = n^2 \int dx''\, \hat{\theta}_{xx''} \Big[\tilde{g}_2(x,t,x',t';x'',t)$$
$$- \frac{1}{V}\, \tilde{g}_2(x,x'',t)\, f_1(x',t')\Big] \, . \tag{22.38}$$

It then follows that the source A obeys the conditions

$$\int dx'\, A(x,t,x',t') = \int dx\, A(x,t,x',t') = 0 \, ,$$

analogous to the property (22.24).

From the expression (22.38) follows that the source $A(x,t,x',t')$ is of order $n\tilde{I} \sim nF_1/\tilde{\tau}_{\mathbf{rel}}$; hence in Eq. (22.34) there are two relaxation times, $\tau_{\mathbf{rel}}, \tilde{\tau}_{\mathbf{rel}}$. For $\tau_{\mathbf{rel}} \ll \tilde{\tau}_{\mathbf{rel}}$, the right-hand side of (22.34) can be neglected. Thus,

$$A(x,t,x',t') = 0 \tag{22.39}$$

and therefore (22.34) is a homogeneous equation:

$$\left(\hat{L}_x + \delta\hat{I}_x\right) \langle \delta N\ \delta N\rangle_{x,t,x',t'}$$
$$+ \langle \delta N\ \delta N\rangle_{\boldsymbol{r},t,x',t'} \cdot \frac{\partial n f_1(x,t)}{\partial \boldsymbol{p}} = 0 \, . \tag{22.40}$$

We arrived at a situation similar to the one prevailing in the description

of the short-range correlations in the polarization approximation. The difference between Eqs (22.21). (22.40) and the corresponding equations (6.13), (6.31) lies only in the substitutions

$$\hat{L}_{xx'} \longrightarrow \hat{L}_{xx'} + \delta\hat{I}_x + \delta\hat{I}_x \ ,$$

$$\hat{L}_x \longrightarrow \hat{L}_x + \delta\hat{I}_x \ .$$

Thus, Eqs (22.21), (22.40) take into account the damping due to the short-range fluctuations, which define the Boltzmann collision integral †.

As in section 6, it is necessary for determining the two-time function $\langle\delta N\ \delta N\rangle_{x,t,x',t'}$ to solve Eq. (22.21) for the one-time correlation. This can be avoided if, instead of the latter, one uses for the definition of the correlations the equation for δN, which is equivalent to the set of Eqs (22.21), (22.40):

$$\left(\hat{L}_x + \delta\hat{I}_x\right)\left\{\delta N(x,t) - \delta N^{\text{source}}(x,t)\right\}$$
$$+ \delta \boldsymbol{F}(\boldsymbol{r},t) \cdot \frac{\partial n f_1(x,t)}{\partial \boldsymbol{p}} = 0 \ . \tag{22.41}$$

The correlation of the source fluctuations is defined by the equation

$$\left(\hat{L}_x + \delta\hat{I}_x\right) \langle \delta N\ \delta N\rangle^{\text{source}}_{x,t,x',t'} = 0 \tag{22.42}$$

which must be solved with the initial condition

$$\langle\delta N\ \delta N\rangle^{\text{source}}_{x,t,x',t'}\Big|_{t=t'} = \left(\hat{L}_{xx'} + \delta\hat{I}_x + \delta\hat{I}_{x'}\right)^{-1} A(x,x',t) \tag{22.43}$$

where $(\dots)^{-1}$ is the inverse operator. The function A is defined by Eqs (22.22), (22,26), (22,23), $(22.27)_1$.

If the contribution of the long-range fluctuations is not taken into account, the last term in (22.41) can be neglected, and in (22.26) for A one may put $\tilde{A} = 0$. From (22.41) follows then $\delta N = \delta N^{\text{source}}$; hence the equation for $\langle\delta N\ \delta N\rangle_{x,t,x',t'}$ coincides with (22.42):

$$\left(\hat{L}_x + \delta\hat{I}_x\right) \langle\delta N\ \delta N\rangle_{x,t,x',t'} = 0 \ . \tag{22.44}$$

Equation (22.44) also follows, of course, from (22.40). Equation (22.21) reduces, in the same approximation, to

$$\left(\hat{L}_{xx'} + \delta\hat{I}_x + \delta\hat{I}_{x'}\right) \langle\delta N\ \delta N\rangle_{x\ x'\ t} = A(x,x',t) \ . \tag{22.45}$$

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† **Equation (22.40) corresponds to Onsager's assumption about the time-evolution of the long-range fluctuations (see footnote on p.26).**

The set of equations (22.44), (22.45) can be replaced by the equivalent Langevin equation

$$\left(\hat{L}_x + \delta\hat{I}_x\right) \delta N(x,t) = y(x,t) \tag{22.48}$$

which represents the linearized Boltzmann equation with a random source $y(x,t)$. Such an equation was first considered by Kadomtsev [25]. He obtained from it expressions for the spectral density of the random source in the equilibrium state.

23. RANDOM SOURCES IN THE BOLTZMANN KINETIC EQUATION AND IN THE HYDRODYNAMICAL EQUATIONS

Let us determine an expression for the spectral density of the source of fluctuations in the linearized Boltzmann equation. We assume that the gas is spatially homogeneous and that $\tilde{I}=0$. Then Eq. (22.8) becomes

$$\frac{\partial f_1}{\partial t} + \boldsymbol{F}_0 \cdot \frac{\partial f_1}{\partial \boldsymbol{p}} = I_B(\boldsymbol{p},t) \tag{23.1}$$

where $\boldsymbol{F}_0$ is the external force.

Using Eq. (22.45) we write an equation for the spatial spectral density $(\delta N\, \delta N)_{\boldsymbol{k},\boldsymbol{p},\boldsymbol{p}',t}$, assuming that the distribution function f_1 changes slowly over an interval $\tau_{\mathbf{rel}}$, the characteristic time of the long-range fluctuations. In zeroth approximation in $\tau_{\mathbf{rel}}(\partial/\partial t)$ we get from (22.45)

$$\left\{i\boldsymbol{k} \cdot (\boldsymbol{v}-\boldsymbol{v}') + \hat{\mathcal{I}}_{\boldsymbol{p}} + \hat{\mathcal{I}}_{\boldsymbol{p}'}\right\}(\delta N\, \delta N)_{\boldsymbol{k},\boldsymbol{p},\boldsymbol{p}',t} = A^{\mathbf{B}}(\boldsymbol{k},\boldsymbol{p},\boldsymbol{p}',t)\,. \tag{23.2}$$

Here, as in references [40,41], we introduce the notation

$$\hat{\mathcal{I}}_{\boldsymbol{p}} = \boldsymbol{F}_0 \cdot \frac{\partial}{\partial \boldsymbol{p}} + \delta\hat{I}_{\boldsymbol{p}}\,. \tag{23.3}$$

The time dependence in Eq. (23.2) is through the functions $f_1(\boldsymbol{p},t)$ and $\boldsymbol{F}_0(t)$.

Using Eq. (22.27), we find an expression for the spatial Fourier components of $A^{\mathbf{B}}(\boldsymbol{r}-\boldsymbol{r}', \boldsymbol{p},\boldsymbol{p}',t)$. To zeroth order in $k l_\phi$ (l_ϕ being a physically infinitesimal length element for the Boltzmann equation, see sect. 16) we obtain from $(22.27)_1$:

$$\begin{aligned} A^{\mathbf{B}}(\boldsymbol{k},\boldsymbol{p},\boldsymbol{p}',t) = n^2 \int d\boldsymbol{r}''\, d\boldsymbol{p}'' \Big\{\hat{\theta}_{xx''} \Big[&\delta(\boldsymbol{p}'-\boldsymbol{p}'') + \delta(\boldsymbol{p}'-\boldsymbol{p}) \\ &- \delta(\boldsymbol{p}'-\boldsymbol{P}'') - \delta(\boldsymbol{p}'-\boldsymbol{P})\Big] f_1(\boldsymbol{P},t)\, f_1(\boldsymbol{P}'',t) \\ &+ \hat{\theta}_{x'x''}\Big[\delta(\boldsymbol{p}-\boldsymbol{p}'') + \delta(\boldsymbol{p}-\boldsymbol{p}') - \delta(\boldsymbol{p}-\boldsymbol{P}'') \\ &- \delta(\boldsymbol{p}-\boldsymbol{P}')\Big]\, f_1(\boldsymbol{P}',t)\, f_1(\boldsymbol{P}'',t)\Big\} \end{aligned} \tag{23.4$_1$}$$

We see that in this approximation the right-hand side is independent of $\boldsymbol{k}$.

Let us perform the corresponding transformation of Eq. $(22.27)_2$. As a result we find an alternative form for the function $A^{\mathbf{B}}$:

$$A^{\mathbf{B}}(\boldsymbol{k},\boldsymbol{p},\boldsymbol{p}',t) = n\left[\delta(\boldsymbol{p}-\boldsymbol{p}')\, I_B(\boldsymbol{p},t) + I_{\boldsymbol{p}\boldsymbol{p}'}\right]$$
$$- n^2 \int d\boldsymbol{r}''\, d\boldsymbol{p}'' \Big\{ \hat{\theta}_{xx''} \left[\delta(\boldsymbol{p}'-\boldsymbol{P}) + \delta(\boldsymbol{p}'-\boldsymbol{P}'')\right] f_1(\boldsymbol{P},t)\, f_1(\boldsymbol{P}'',t)$$
$$+ \hat{\theta}_{x'x''}\left[\delta(\boldsymbol{p}-\boldsymbol{P}') + \delta(\boldsymbol{p}-\boldsymbol{P}'')\right] f_1(\boldsymbol{P}',t)\, f_1(\boldsymbol{P}'',t)\Big\} \qquad (23.4)_2$$

where $I_{\boldsymbol{p}\boldsymbol{p}'} = \int d(\boldsymbol{r}-\boldsymbol{r}') I_{xx'}$, the function $I_{xx'}$ being defined by (22.29).

Thus the spectral density $(\delta N\, \delta N)_{\boldsymbol{k},\boldsymbol{p},\boldsymbol{p}',t}$ is defined by Eq. (23.2) with the right-hand side $(23.4)_1$ or $(23.4)_2$.

We now write the corresponding equation for the space-time spectral density $(\delta N\, \delta N)_{\omega,\boldsymbol{k},\boldsymbol{p},\boldsymbol{p}',t}$ in zeroth approximation with respect to $\tau_{\mathbf{rel}}(\partial/\partial t)$ from (22.44):

$$(\delta N\, \delta N)^{(+)}_{\omega,\boldsymbol{k},\boldsymbol{p},\boldsymbol{p}',t} \equiv \int_0^\infty d(t-t')(\delta N\, \delta N)_{t-t',\boldsymbol{k},\boldsymbol{p},\boldsymbol{p}',t}\, e^{i\omega(t-t')}$$
$$= \frac{(\delta N\, \delta N)_{\boldsymbol{k},\boldsymbol{p},\boldsymbol{p}',t}}{-i(\omega-\boldsymbol{k}\cdot\boldsymbol{v}) + \hat{I}_{\boldsymbol{p}}}$$

We then find:

$$(\delta N\, \delta N)_{\omega,\boldsymbol{k},\boldsymbol{p},\boldsymbol{p}',t} = \frac{[i(\boldsymbol{k}\cdot\boldsymbol{v}-\boldsymbol{k}\cdot\boldsymbol{v}') + \hat{I}_p + \hat{I}_{p'}](\delta N\, \delta N)_{\boldsymbol{k},\boldsymbol{p},\boldsymbol{p}',t}}{[-i(\omega-\boldsymbol{k}\cdot\boldsymbol{v}) + \hat{I}_{\boldsymbol{p}}]\,[-i(\omega-\boldsymbol{k}\cdot\boldsymbol{v}') + \hat{I}_{\boldsymbol{p}'}]} \qquad (23.5)$$

which expresses the space-time spectral density in terms of the spatial spectral density. Using Eq. (23.2) we can rewrite (23.5) in the form

$$(\delta N\, \delta N)_{\omega,\boldsymbol{k},\boldsymbol{p},\boldsymbol{p}',t} = \frac{A^{\mathbf{B}}(\boldsymbol{k},\boldsymbol{p},\boldsymbol{p}',t)}{[-i(\omega-\boldsymbol{k}\cdot\boldsymbol{v}) + \hat{I}_{\boldsymbol{p}}]\,[-i(\omega-\boldsymbol{k}\cdot\boldsymbol{v}') + \hat{I}_{\boldsymbol{p}'}]}\,. \qquad (23.6)$$

Thus, we succeeded in expressing the space-time spectral density of the fluctuations of the smoothed phase density through the function $A^{\mathbf{B}}(\boldsymbol{k},\boldsymbol{p},\boldsymbol{p}',t)$ which, in turn, is expressed in terms of $f_1(\boldsymbol{p},t)$ by Eqs $(23.4)_1$, $(23.4)_2$.

At this point it is not difficult to find the spectral density of the random source in the linearized Boltzmann equation (22.46). From Eq. (22.46) we find an expression interrelating the densities $(\delta N\, \delta N)_{\omega,\boldsymbol{k},\boldsymbol{p},\boldsymbol{p}',t}$ and $(yy)_{\omega,\boldsymbol{k},\boldsymbol{p},\boldsymbol{p}',t}$:

$$(\delta N\, \delta N)_{\omega,\boldsymbol{k},\boldsymbol{p},\boldsymbol{p}',t} = \frac{(yy)_{\omega,\boldsymbol{k},\boldsymbol{p},\boldsymbol{p}',t}}{[-i(\omega-\boldsymbol{k}\cdot\boldsymbol{v}) + \hat{I}_{\boldsymbol{p}}]\,[-i(\omega-\boldsymbol{k}\cdot\boldsymbol{v}') + \hat{I}_{\boldsymbol{p}'}]}\,. \qquad (23.7)$$

Equating expressions (23.6) and (23.7) we find

$$(yy)_{\omega,\boldsymbol{k},\boldsymbol{p},\boldsymbol{p}',t} = A^{\mathbf{B}}(\boldsymbol{p},\boldsymbol{p}',t)\,. \qquad (23.8)$$

Here we took into account the fact that, in zeroth approximation in $k\,l_\phi$, the function $A^{\mathbf{B}}(\boldsymbol{k},\boldsymbol{p},\boldsymbol{p}',t)$ is independent of $\boldsymbol{k}$.

Thus, the spectral density of the random source in the Boltzmann equation is defined by Eqs $(23.4)_1$ or $(23.4)_2$.

As the right-hand side of Eq. (23.8) is independent of ω and of $\boldsymbol{k}$, the space-time correlation of y is given by

$$(yy)_{\tau,\boldsymbol{r}-\boldsymbol{r}',\boldsymbol{p},\boldsymbol{p}',t} = A^{\mathrm{B}}(\boldsymbol{p},\boldsymbol{p}',t)\,\delta(\tau)\,\delta(\boldsymbol{r}-\boldsymbol{r}')\,. \tag{23.9}$$

Thus the random source, in zeroth approximation with respect to kl_ϕ and to the retardation, is δ-correlated in space and time.

In the equilibrium state the first two terms in the expression $(23.4)_2$ tend to zero and the spectral density $(yy)_{\omega,\boldsymbol{k},\boldsymbol{p},\boldsymbol{p}'}$ is defined by the last term on the right-hand side of $(23.4)_2$, where f_1 is the Maxwell distribution. Using the definition of the operator $\delta\hat{I}_{\boldsymbol{p}}$, (22.37), this spectral density can be rewritten as follows in equilibrium:

$$(yy)_{\omega,\boldsymbol{k},\boldsymbol{p},\boldsymbol{p}'} = (\delta\hat{I}_{\boldsymbol{p}} + \delta\hat{I}_{\boldsymbol{p}'})\; n\,\delta(\boldsymbol{p}-\boldsymbol{p}')\,f_1(\boldsymbol{p})\,. \tag{23.10}$$

This result was first obtained in Kadomtsev's paper [25].

In a non-equilibrium state, the following equation can be used instead of (23.8), $(23.4)_2$:

$$(yy)_{\omega,\boldsymbol{k},\boldsymbol{p},\boldsymbol{p}',t} = \delta(\boldsymbol{p}-\boldsymbol{p}')\,n\,I_B(\boldsymbol{p},t) + n\,I_{\boldsymbol{p}\boldsymbol{p}'}(t)$$
$$+ (\delta\hat{I}_{\boldsymbol{p}} + \delta\hat{I}_{\boldsymbol{p}'})\; n\,\delta(\boldsymbol{p}-\boldsymbol{p}')\,f_1(\boldsymbol{p},t)\,, \tag{23.11}$$

where

$$\hat{I}_{\boldsymbol{p}\boldsymbol{p}'} = n\int d(\boldsymbol{r}-\boldsymbol{r}')\,\hat{\theta}_{xx'}\,f_1(\boldsymbol{P},t)\,f_1(\boldsymbol{P}',t)\,. \tag{23.12}$$

Following the terminology of ref. [40], this expression might be called 'the incomplete collision integral'. because $\int d\boldsymbol{p}'\,I_{\boldsymbol{p}\boldsymbol{p}'} = I_B(\boldsymbol{p},t)$.

In a steady state, when the function f_1 is independent of time, Eq. (23.11) coincides with the one obtained by Gantsevich, Gurevich and Katilius [40,41].

In the work of Gor'kov, Dzyaloshinsky and Pitaevski [38] the spectral density (23.10) is expressed through the transition probabilities. The corresponding result for a non-equilibrium stationary state was obtained in the work of Kogan and Shul'man [39].

We note again that Eqs $(23.4)_1$, $(23.4)_2$, (23.11) are only valid under the condition that the contribution of the long-range fluctuations to the kinetic equation (21.1) are negligible, i.e. $\tilde{I}=0$. In that approximation, the terms in δF, describing the long-range fluctuations of the force, can be neglected in Eqs (22.40), (22.41). These limitations will be lifted in Chapter 11, treating the kinetic theory of fluctuations in a plasma.

We consider now the general properties of the spectral density (23.8), which are analogous to the properties (11.5), (11.7) of the Boltzmann collision

integral. For the function $A^{\mathrm{B}}(\boldsymbol{p},\boldsymbol{p}',t)$ we use Eq. $(23.4)_2$.

We multiply that function by $\phi(\boldsymbol{p})$, $\psi(\boldsymbol{p}')$ and integrate over $\boldsymbol{p},\boldsymbol{p}'$. We introduce the quantity

$$I_{\omega,\boldsymbol{k}}(t) = \int d\boldsymbol{p}_1\, d\boldsymbol{p}_2\, \phi(\boldsymbol{p}_1)\, \psi(\boldsymbol{p}_2)\, (yy)_{\omega,\boldsymbol{k},\boldsymbol{p}_1,\boldsymbol{p}_2,t}\,, \tag{23.13}$$

After symmetrization in $\boldsymbol{p}_1,\boldsymbol{p}_2$ we obtain :

$$\begin{aligned} I_{\omega,\boldsymbol{k}}(t) = \frac{n^2}{2}\int d\boldsymbol{r}_2\, d\boldsymbol{p}_1\, d\boldsymbol{p}_2\, [\phi(\boldsymbol{p}_1)+\phi(\boldsymbol{p}_2)][\psi(\boldsymbol{p}_1)+\psi(\boldsymbol{p}_2)] \\ \hat{\theta}_{12} f_1(\boldsymbol{P}_1,t)\, f_1(\boldsymbol{P}_2,t) - \frac{n^2}{2}\int d\boldsymbol{r}_2\, d\boldsymbol{p}_1\, d\boldsymbol{p}_2 \Big\{[\phi(\boldsymbol{p}_1)+\phi(\boldsymbol{p}_2)] \\ \hat{\theta}_{12}[\psi(\boldsymbol{P}_1)+\psi(\boldsymbol{P}_2)]\, f_1(\boldsymbol{P}_1,t)\, f_1(\boldsymbol{P}_2,t) + (\phi \rightleftharpoons \psi)\Big\}\,. \end{aligned} \tag{23.14}$$

Here we used the expression for the Boltzmann collision integral I^{B} and Eq. (23.12) for the 'incomplete collision integral'.

The form (23.14) corresponds to the Bogolyubov representation. For finding the general properties of (23.14) it is more convenient to use the Boltzmann form. We therefore substitute in (23.14):

$$\hat{\theta}_{12} = \frac{\partial \Phi_{12}}{\partial \boldsymbol{r}_1}\cdot\frac{\partial}{\partial \boldsymbol{p}_1} + \frac{\partial \Phi_{12}}{\partial \boldsymbol{r}_2}\cdot\frac{\partial}{\partial \boldsymbol{p}_2} \longrightarrow \boldsymbol{v}_1\cdot\frac{\partial}{\partial \boldsymbol{r}_1} + \boldsymbol{v}_2\cdot\frac{\partial}{\partial \boldsymbol{r}_2}\,. \tag{23.15}$$

As in the passage from (10.10) to (10.18), we use the cylindrical coordinate system z,ρ,α with the z-axis along the relative velocity vector $\boldsymbol{v}_1-\boldsymbol{v}_2$. Equation (23.14) takes the form

$$\begin{aligned} I_{\omega,\boldsymbol{k}}(t) = \frac{n^2}{2}\int_{-\infty}^{\infty} dz \int_0^{\infty} d\rho\,\rho \int_0^{2\pi} d\alpha \int d\boldsymbol{p}_1\, d\boldsymbol{p}_2\, |\boldsymbol{v}_2-\boldsymbol{v}_1| \Big\{[\phi(\boldsymbol{p}_1)+\phi(\boldsymbol{p}_2)] \\ \frac{\partial}{\partial t}\ \tfrac{1}{2}[\psi(\boldsymbol{p}_1)+\psi(\boldsymbol{p}_2)-\psi(\boldsymbol{P}_1)+\psi(\boldsymbol{P}_2)] f_1(\boldsymbol{P}_1,t)\, f_1(\boldsymbol{P}_1,t) \\ + (\phi \rightleftharpoons \psi)\Big\}\,. \end{aligned} \tag{23.16}$$

We integrate over z and, as in section 10, consider two types of collisions. For $z=-\infty$, $\boldsymbol{P}_{1,2}\to\boldsymbol{p}_{1,2}$ (see (10.16)), and $\boldsymbol{p}'_{1,2}$ are the momenta after the collisions of type a. For $z=+\infty$, $P_{1,2}\to\boldsymbol{p}'_{1,2}$ and $\boldsymbol{p}_{1,2}$ are the momenta after the collisions of type b (10.17). We thus obtain

$$\begin{aligned} I_{\omega,\boldsymbol{k}}(t) = \frac{n^2}{2}\int_0^{\infty} d\rho\,\rho \int_0^{2\pi} d\alpha \int d\boldsymbol{p}_1\, d\boldsymbol{p}_2\, |\boldsymbol{v}_2-\boldsymbol{v}_1|\,\cdot \\ \Big\{\tfrac{1}{2}\Big[\Big(\phi(\boldsymbol{p}_1)+\phi(\boldsymbol{p}_2)\Big)\Big(\psi(\boldsymbol{p}_1)+\psi(\boldsymbol{p}_2)\Big) f_1(\boldsymbol{p}'_1,t)\, f_1(\boldsymbol{p}'_2,t) \\ -\Big(\phi(\boldsymbol{p}_1)+\phi(\boldsymbol{p}_2)\Big)\Big(\psi(\boldsymbol{p}_1)+\psi(\boldsymbol{p}_2)\Big) f_1(\boldsymbol{p}_1,t)\, f_1(\boldsymbol{p}_2,t)\Big] \end{aligned}$$

$$- \Big[\Big(\phi(\boldsymbol{p}_1) + \phi(\boldsymbol{p}_2)\Big)\Big(\psi(\boldsymbol{p}_1') + \psi(\boldsymbol{p}_2')\Big) f_1(\boldsymbol{p}_1', t)\, f_1(\boldsymbol{p}_2', t)$$
$$- \Big(\phi(\boldsymbol{p}_1) + \phi(\boldsymbol{p}_2)\Big)\Big(\psi(\boldsymbol{p}_1) + \psi(\boldsymbol{p}_2)\Big) f_1(\boldsymbol{p}_1, t)\, f_1(\boldsymbol{p}_2, t)\Big]$$
$$+ (\phi \rightleftharpoons \psi)\Big\} .$$

We now transform the expression enclosed in curly brackets. After reduction and symmetrization with respect to $\boldsymbol{p}_1, \boldsymbol{p}_2$ and $\boldsymbol{p}_1', \boldsymbol{p}_2'$ we get

$$\Big\{\ \Big\} = \tfrac{1}{2}\Big\{\Big[\phi(\boldsymbol{p}_1) + \phi(\boldsymbol{p}_2)\Big]\Big[\psi(\boldsymbol{p}_1) + \psi(\boldsymbol{p}_2)\Big]$$
$$- 2\Big[\phi(\boldsymbol{p}_1') + \phi(\boldsymbol{p}_2')\Big]\Big[\psi(\boldsymbol{p}_1) + \psi(\boldsymbol{p}_2)\Big]$$
$$+ \Big[\phi(\boldsymbol{p}_1') + \phi(\boldsymbol{p}_2')\Big]\Big[\psi(\boldsymbol{p}_1') + \psi(\boldsymbol{p}_2')\Big] + (\phi \rightleftharpoons \psi)\Big\} f_1(\boldsymbol{p}_1, t)\, f_1(\boldsymbol{p}_2, t)$$
$$= \Big[\phi(\boldsymbol{p}_1) + \phi(\boldsymbol{p}_2) - \phi(\boldsymbol{p}_1') - \phi(\boldsymbol{p}_2')\Big]\Big[\psi(\boldsymbol{p}_1) + \psi(\boldsymbol{p}_2) - \psi(\boldsymbol{p}_1') - \psi(\boldsymbol{p}_2')\Big]$$
$$\cdot f_1(\boldsymbol{p}_1, t)\, f_1(\boldsymbol{p}_2, t) .$$

As a result

$$I_{\omega,\boldsymbol{k}}(t) = \frac{n^2}{2}\int_0^\infty d\rho\, \rho \int_0^{2\pi} d\alpha \int d\boldsymbol{p}_1\, d\boldsymbol{p}_2\, |\boldsymbol{v}_2 - \boldsymbol{v}_1|\, f_1(\boldsymbol{p}_1, t)\, f_1(\boldsymbol{p}_2, t)$$
$$\cdot \Big[\phi(\boldsymbol{p}_1) + \phi(\boldsymbol{p}_2) - \phi(\boldsymbol{p}_1') - \phi(\boldsymbol{p}_2')\Big]\Big[\psi(\boldsymbol{p}_1) + \psi(\boldsymbol{p}_2) - \psi(\boldsymbol{p}_1') - \psi(\boldsymbol{p}_2')\Big] . \tag{23.17}$$

From this expression follow immediately the properties:

$$I_{\omega,\boldsymbol{k}}(t) = 0 \quad \begin{array}{l} \text{for } \phi = 1, \boldsymbol{p}, p^2/2m, \text{ and arbitrary } \psi \\ \text{for } \psi = 1, \boldsymbol{p}, p^2/2m, \text{ and arbitrary } \phi \end{array} \tag{23.18$_1$}$$

and

$$I_{\omega,\boldsymbol{k}}(t) \geqslant 0 \ \text{ for } \phi = \psi, \text{ in particular for } \phi = \psi = -k_B \ln f_1(\boldsymbol{p}, t) \tag{23.18$_2$}$$

The equality is valid only in the equilibrium state.

Hence, in going over from the Boltzmann equation for the random function N with the source y:

$$\left(\frac{\partial}{\partial t} + \boldsymbol{v}\cdot\frac{\partial}{\partial \boldsymbol{r}} + \boldsymbol{F}\cdot\frac{\partial}{\partial \boldsymbol{p}}\right) N(x, t)$$
$$= \int dx'\, \hat{\theta}_{xx'}\, N(\boldsymbol{r}, \boldsymbol{P}, t)\, N(\boldsymbol{r}, \boldsymbol{P}', t) + y , \tag{23.19}$$

to the hydrodynamical equations for the random functions †

$$\rho(\boldsymbol{r}, t) = m\int d\boldsymbol{p}\, N , \qquad \rho\boldsymbol{u} = \int d\boldsymbol{p}\, \boldsymbol{p}\, N , \qquad W = \int d\boldsymbol{p}\, \frac{p^2}{2m} N . \tag{23.20}$$

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† Upon averaging, these functions go over into (7.1).

we can see that

$$m\int d\boldsymbol{p}\, y(\boldsymbol{r},\boldsymbol{p},t) = 0\,, \quad \int d\boldsymbol{p}\,\boldsymbol{p}\, y(\boldsymbol{r},\boldsymbol{p},t) = 0\,, \quad \int d\boldsymbol{p}\,\frac{p^2}{2m}\, y(\boldsymbol{r},\boldsymbol{p},t) = 0\,. \tag{23.21}$$

Hence the y does not enter explicitly the equations of balance of the particle density, of the momentum density, or of the kinetic energy density. But the transport equations are not deterministic, because they involve the functions π_{ij}, S_i, i.e. the viscous pressure tensor and the heat flow vector (see Eqs (15.27) - (15.29)). Calculating these functions, e.g. by the Grad method, we find the expressions

$$\pi_{ij} = -\eta\left(\frac{\partial u_i}{\partial r_j} + \frac{\partial u_j}{\partial r_i} - \tfrac{2}{3}\,\delta_{ij}\,\frac{\partial}{\partial r_k}\,u_k\right) + \delta\pi_{ij}$$

$$S_i = -\kappa\,\frac{\partial T}{\partial r_i} + \delta S_i\,, \tag{23.22}$$

which differ from Eqs. (7.17), (7.18) by the presence of the random sources $\delta\pi_{ij}$, δS_i. The spectral density of these sources are expressed in terms of the spectral density of the source y in the Boltzmann equation (23.19):

$$\left(\delta\pi_{ij}\,\delta\pi_{kl}\right)_{\omega,\boldsymbol{k}} = m^2\int d\boldsymbol{p}_1\, d\boldsymbol{p}_2\,\delta v_{1i}\,\delta v_{1j}\,\delta v_{2k}\,\delta v_{2l}(y\,y)_{\omega,\boldsymbol{k},\boldsymbol{p}_1,\boldsymbol{p}_2}$$

$$\left(\delta S_i\,\delta S_j\right)_{\omega,\boldsymbol{k}} = \int d\boldsymbol{p}_1\, d\boldsymbol{p}_2\,\delta_{1i}\;\;\delta v_{2j}\,\frac{m\delta v_1^2}{2}\,\frac{m\delta v_2^2}{2}\,(y\,y)_{\omega,\boldsymbol{k},\boldsymbol{p}_1,\boldsymbol{p}_2} \tag{23.23}$$

where $\delta\boldsymbol{v} = \boldsymbol{v} - \boldsymbol{u}$ is the peculiar velocity.

In Grad's thirteen-moment approximation, Eqs (23.23) reduce to

$$\left(\delta\pi_{ij}\,\delta\pi_{kl}\right)_{\omega,\boldsymbol{k}} = 2\eta k_B T\left[\delta_{il}\,\delta_{jk} + \delta_{ik}\,\delta_{jl} - \tfrac{2}{3}\,\delta_{ij}\,\delta_{kl}\right]\,, \tag{23.24}$$

$$\left(\delta S_i\,\delta S_j\right)_{\omega,\boldsymbol{k}} = 2\kappa k_{\mathrm{B}} T^2 \delta_{ij}\,. \tag{23.25}$$

These expressions are identical to those obtained phenomenologically by Landau and Lifshitz [60].

Thus in the hydrodynamical equations for the ideal gas, the random sources enter the expressions of the viscous pressure tensor and of the heat flow vector (23.22). For nonideal gases these statements might be modified, because the properties of the function $I_{\omega,\boldsymbol{k}}(t)$ are different.

The property $(23.18)_2$ is important for the study of the evolution of the entropy fluctuations.

In concluding this chapter, we note that when the contribution of the long-range fluctuations are retained in the Boltzmann equation, i.e., for $\tilde{I} = 0$, there appear new dissipative terms in the hydrodynamical equations. These are corrections to the viscosity and heat conduction coefficients due to the long range fluctuations. As a result there exist additional terms in the expressions (23.24), (23.25) for the spectral density of the sources $\delta\pi_{ij}$, δS_i.

PART II

Kinetic Theory of Nonideal Fully Ionized Plasmas

INTRODUCTION

In gases, the fundamental parameter is the density parameter $\varepsilon = nr_0^3$. Whenever this parameter is small, it is possible to replace the hierarchy of equations for the distribution functions $f_1, f_2, \ldots$ by a closed kinetic equation for the one-particle distribution function f_1. To first order in the density parameter, i.e. in the binary collision approximation, this equation is the Boltzmann equation.

When $\varepsilon \ll 1$, the number of particles in a volume r_0^3 is much less than unity. In a plasma, the situation is different. The role of the range of the effective interactions of the charged particles is played by the Debye radius r_{D}, defined as

$$r_{\mathrm{D}}^2 = k_B T \Big/ \left(4\pi \sum_a e_a^2 n_a \right) .$$

In most plasmas, the number of particles in a Debye sphere is very large, i.e. $nr_{\mathrm{D}}^3 \gg 1$. This means that each particle interacts simultaneously with a large number of other particles. The smallness parameter for plasmas is thus

$$\mu = \frac{1}{nr_{\mathrm{D}}^3}$$

which is called the plasma parameter.

From the definitions of the quantities μ and r_{D} follows that

$$\mu \sim \frac{e^2}{r_{\mathrm{D}} k_B T}$$

i.e., the plasma parameter is of the order of the ratio of the interaction energy at distance r_{D} to the kinetic energy.

Along with μ, another parameter appears in the theory of plasmas:

$$\xi = \frac{e^2}{r_{\mathrm{av}} k_B T} \sim \mu \frac{r_{\mathrm{D}}}{r_{\mathrm{av}}} \sim \mu^{\frac{2}{3}}$$

the interaction parameter. It is defined as the ratio of the interaction energy at the distance r_{av} to the kinetic energy. For $\xi \ll 1$, the role of the interactions is small, the plasma is close to ideal $(\xi \ll 0)$, and the plasma can be considered as weakly coupled. From the definition of the parameters follows that whenever $r_{\mathrm{av}} < r_{\mathrm{D}}$, we have $\mu < \xi$.

The possibility of obtaining a kinetic equation for the plasma is not due to the smallness of the density parameter as in gases, but to the fact that each particle interacts simultaneously with many others. As a result, the density fluctuations (as well as the electric and magnetic field fluctuations) in any volume element, large compared to r_{av}^3, are small. The zeroth approximation in the plasma parameter corresponds to the neglect of these fluctuations.

The modern kinetic theory of plasmas is based on the work of L.D. Landau (1936), A.A. Vlasov (1938) and N.N. Bogolyubov (1946). Landau derived from the Boltzmann equation a kinetic equation for charged particles. The derivation of the Landau equation is based on a perturbation expansion in the parameter ξ . However, perturbation theory is not sufficient for the description of a plasma. This becomes evident by noting that the collision integral in the Landau equation diverges logarithmically.

The divergence at short distances (large wave-vectors) cannot be cured in the framework of perturbation theory, because at short distances the interaction energy is not small. To remove this divergence, one must sum an infinite perturbation series. For a dilute plasma, the result obtained in this way corresponds to the binary collision approximation.

The divergence of the collision integral at large distances is due to the fact that in using perturbation theory (to lowest non-trivial order in ξ), one does not take into account the collective character of the charged particle interactions, i.e., the fact that each particle interacts simultaneously with many others. As a result, the Landau collision integral does not display the plasma polarization effects.

In order to construct a kinetic theory of plasmas, it is more appropriate to use μ as a perturbation parameter. This corresponds to a perturbation theory in the fluctuations, or more precisely, in the correlation functions.

The zeroth approximation corresponds to the complete neglect of the correlations. In this approximation (section 29) one obtains a self-consistent set of equations for the one-particle distribution functions f_a (where a is the index labelling for the species of particles) coupled to the Maxwell equations for the average electric and magnetic fields. The charge and current densities in the

Maxwell equations are defined in terms of the distribution functions f_a. These equations were first derived by A.A. Vlasov in 1938.

In the Vlasov equations the correlation effects are completely neglected. As a result, they do not describe dissipative processes, and conserve the total entropy of the plasma.

In Bogolyubov's book 'Problems of dynamical theory in statistical physics' it was shown that both the Vlasov and the Landau equations can be obtained as particular approximations of the hierarchy for the distribution functions, f_a, f_{ab}, f_{abc}, ... of the charged particles.

The derivation of these equations is based on a perturbation expansion in powers of ξ. The Vlasov equation corresponds to the first approximation, and the Landau equation to the second approximation in ξ. In order to take into account collective effects, Bogolyubov introduced a perturbation expansion with respect to the plasma parameter (the polarization approximation). To first order in μ he obtained a closed set of equations for the one-particle function f_a and the two-particle correlations g_{ab}. The equation for the latter differs from the corresponding equation in the binary collision approximation by the presence of additional terms, accounting for the plasma polarization.

In the work of Lenard and Balescu the equation for g_{ab} was solved, with the assumption of complete weakening of the initial correlations and the neglect of the retardation effects and of the spatial inhomogeneity. As a result, they obtained an expression for the collision integral taking into account the polarization of the plasma (sect. 37) [5,6].

The Landau and the Balescu-Lenard kinetic equations take into account the effect of the interactions only in the dissipative characteristics of the plasma. In that sense, they are kinetic equations for ideal plasmas. In Chapters 9 and 10, generalized kinetic equations will be derived. These yield the complete contribution of the interactions, to first order in μ. These are kinetic equations for the nonideal plasma.

In refs [1,4-6] the kinetic equations were derived only for the case of Coulomb interactions (Coulomb plasma). The Bogolyubov method can be used also in more general cases, in which the complete electromagnetic interactions are retained. However, this method becomes very complicated for the description of such a plasma.

In these cases it is more convenient to use as a starting point the set of equations for the microscopic phase densities of each plasma component, $N_a(x,t)$ and the microscopic electric and magnetic fields (sect. 24). The use of this method simplifies the solution of many problems. This is due to the fact that, instead of solving complicated equations for the distribution functions and for the field quantities, one is confronted with simpler equations for the moments of the phase

densities and of the microscopic fields. Even simpler is the use of the equations for the random deviations of these functions from their average value (sects 34,35).

The Landau equation and the Balescu-Lenard equation have a common insufficiency. In both collision integrals there appears a divergence at large values of the wave-number. This corresponds to short distances between the particles. On the contrary, if the Boltzmann equation is used (in which ξ is not supposed to be small) the collision integral diverges at large distances.

For a description of the non-equilibrium processes in a plasma it is necessary to possess a kinetic equation which correctly describes the interactions of the charged particles at both long and short distances. As a result, there appears the problem of deriving a generalized kinetic equation for the plasma, taking into account both the collective interactions and the binary collisions.

The possibility of the construction of such a theory is shown in sect. 56.

CHAPTER 5

The Microscopic Equations for a fully Ionized Plasma and their Average

24. MICROSCOPIC EQUATIONS FOR A FULLY IONIZED PLASMA

In the first part, two methods were considered for the description of the non-equilibrium processes in gases. One of them took as a starting point the Liouville equation for the distribution function of the positions and momenta of all the particles $f_N(x,t)$. From the Liouville equation a hierarchy can be derived for the reduced distribution functions $f_1, f_2, \ldots$ The approximate solution of this hierarchy under specific conditions leads to the kinetic equations for gases, i.e. closed equations for the one-particle distribution function f_1.

The second method takes as a starting point the equations for the phase density in the six-dimensional space of position and momentum, i.e. for the functions

$$N(x,t) = \sum_{i=1}^{N} \delta[x - x_i(t)] , \qquad x = (\boldsymbol{r}, \boldsymbol{p}) . \tag{24.1}$$

The evolution of the phase densities is defined by Eqs (4.6) or (4.7).

From the equations for the phase density, a hierarchy can be derived for the moments of this density. This hierarchy is equivalent to the one for the functions $f_1, f_2, \ldots$.

The state of a fully ionized plasma is defined not only by the values of the coordinates and of the momenta of all the particles, but also by the values of the microscopic amplitudes of the electric and magnetic field in every point of space. Therefore, the description of non-equilibrium processes on the basis of the Liouville equation for the distribution function of the particle variables and of the field is very complicated. It is more convenient to use the equations for the microscopic phase densities of all the particles and the microscopic fields.

We denote by e_a, m_a the charge and the mass of a particle of species a. In a fully ionized electron-ion plasma the index a takes two values :

$$a = e, i \quad (e_e \equiv e\,,\; e_i = Z|e|)\,.$$

N_a denotes the total number of particles of species a; $N_a/V \equiv n_a$ is the average concentration of particles of species a.

The condition of electroneutrality of the plasma can be written in the form

$$\sum_a e_a N_a = 0 \qquad \text{or} \qquad \sum_a e_a n_a = 0\,. \tag{24.2}$$

The phase-density of particles of species a in the corresponding six-dimensional space $x = (\boldsymbol{r}, \boldsymbol{p})$ is defined by

$$N_a(\boldsymbol{r}, \boldsymbol{p}, t) = \sum_{i=1}^{N_a} \delta[\boldsymbol{r} - \boldsymbol{r}_{ia}(t)]\; \delta[\boldsymbol{p} - \boldsymbol{p}_{ia}(t)] \tag{24.3}$$

or, more briefly

$$N_a(x,t) = \sum_{i=1}^{N_a} \delta[x - x_{ia}(t)]\,. \tag{24.4}$$

From these definitions follows that $N_a(x,t)dx$ equals the number of particles of species a, the positions and momenta of which are in a volume element dx around x at time t. The integral over the whole phase space equals the total number of particles of species a:

$$\int dx\, N_a(x,t) = N_a\,.$$

The evolution of the phase density $N_a(x,t)$ is determined by an equation analogous to (4.6): $[\boldsymbol{v} = \boldsymbol{p}/m_a]$

$$\frac{\partial N_a}{\partial t} + \boldsymbol{v} \cdot \frac{\partial N_a}{\partial \boldsymbol{r}} + \boldsymbol{F}_a^{\mathrm{M}}(\boldsymbol{r}, t) \cdot \frac{\partial N_a}{\partial \boldsymbol{p}} = 0\,. \tag{24.5}$$

The force $\boldsymbol{F}_a^{\mathrm{M}}$ is defined as follows

$$\boldsymbol{F}_a^{\mathrm{M}} = \boldsymbol{F}_{a0} + e_a \boldsymbol{E}^{\mathrm{M}}(\boldsymbol{r}, t) + \frac{e_a}{c}\, [\boldsymbol{v} \times \boldsymbol{B}^{\mathrm{M}}(\boldsymbol{r}, t)] \tag{24.6}$$

where $\boldsymbol{F}_{a0}$ is the external force acting on the charged particle, and $\boldsymbol{E}^{\mathrm{M}}, \boldsymbol{B}^{\mathrm{M}}$ are the amplitudes of the microscopic electric and magnetic fields.

The microscopic charge and current density $q^{\mathrm{M}}, \boldsymbol{j}^{\mathrm{M}}$ are expressed in terms of the phase density:

$$\begin{aligned} q^{\mathrm{M}}(\boldsymbol{r}, t) &= \sum_a e_a \int d\boldsymbol{p}\; N_a(\boldsymbol{r}, \boldsymbol{p}, t) \\ \boldsymbol{j}^{\mathrm{M}}(\boldsymbol{r}, t) &= \sum_a e_a \int d\boldsymbol{p}\; \boldsymbol{v}\, N_a(\boldsymbol{r}, \boldsymbol{p}, t) \end{aligned} \tag{24.7}$$

The equations for the microscopic fields, i.e. the Maxwell equations, are then:

$$\nabla \times \boldsymbol{B}^{\mathrm{M}} = \frac{1}{c}\frac{\partial \boldsymbol{E}^{\mathrm{M}}}{\partial t} + \frac{4\pi}{c}\sum_a e_a \int d\boldsymbol{p}\, \boldsymbol{v}\, N_a(\boldsymbol{r}, \boldsymbol{p}, t) \tag{24.8}$$

$$\nabla \times \boldsymbol{E}^{\mathrm{M}} = -\frac{1}{c}\frac{\partial \boldsymbol{B}^{\mathrm{M}}}{\partial t} \tag{24.9}$$

$$\nabla \cdot \boldsymbol{B}^{\mathrm{M}} = 0 \tag{24.10}$$

$$\nabla \cdot \boldsymbol{E}^{\mathrm{M}} = 4\pi \sum_a e_a \int d\boldsymbol{p}\, N_a(\boldsymbol{r}, \boldsymbol{p}, t)\ . \tag{24.11}$$

Equations (24.5), (24.6), (24.8)-(24.11) are a closed set of equations for the microscopic functions $N_a(x,t)$, $\boldsymbol{E}^{\mathrm{M}}(\boldsymbol{r},t)$, $\boldsymbol{B}^{\mathrm{M}}(\boldsymbol{r},t)$.

Instead of the equations for the functions $\boldsymbol{E}^{\mathrm{M}}, \boldsymbol{B}^{\mathrm{M}}$ we can also use a set of equations for the potentials $\boldsymbol{A}^{\mathrm{M}}, \phi^{\mathrm{M}}$:

$$\boldsymbol{B}^{\mathrm{M}} = \nabla \times \boldsymbol{A}$$

$$\boldsymbol{E}^{\mathrm{M}} = -\nabla \phi^{\mathrm{M}} - \frac{1}{c}\frac{\partial \boldsymbol{A}^{\mathrm{M}}}{\partial t}\ . \tag{24.12}$$

The well-known arbitrariness in the choice of the potentials requires the formulation of additional conditions.

The microscopic equations (24.5), (24.6), (24.8)-(24.11) can be used for relativistic plasmas as well, provided we change the relation between momentum and velocity $\boldsymbol{p} = m_a \boldsymbol{v}$, into the relativistic relation

$$\boldsymbol{p} = m_a \boldsymbol{v}(1 - v^2/c^2)^{-\frac{1}{2}} \equiv \gamma m_a \boldsymbol{v}\ . \tag{24.13}$$

In fact,

$$N_a(\boldsymbol{r}, \boldsymbol{p}, t)\ d\boldsymbol{r}\, d\boldsymbol{p} \tag{24.14}$$

defines the number of particles of species a in the volume element $d\boldsymbol{r}\,d\boldsymbol{p}$ and does not depend on the choice of the reference frame. As the phase volume element $d\boldsymbol{r}\,d\boldsymbol{p}$ is invariant, the function $N_a(\boldsymbol{r}, \boldsymbol{p}, t)$ is therefore also invariant under a change of reference frame (more details are found in refs [8,9]).

For the description of a relativistic plasma one may also use, instead of $N_a(\boldsymbol{r}, \boldsymbol{p}, t)$, a phase density in an eight-dimensional space spanned by the four-position $Q = (\boldsymbol{r}, ict)$ and the four-momentum $P = (\boldsymbol{p}, i\mathcal{E}/c)$. The function $N_a(Q,P)$ is defined by

$$N_a(Q,P) = \sum_{i=1}^{N_a} \int ds_i\, \delta[Q - Q_{ia}(s_i)]\ \delta[P - P_{ia}(s_i)] \tag{24.15}$$

where s_i is the proper time of particle i, of species a.

Instead of Eqs (24.5), (24.6), (24.8)-(24.11) we may then use a set of equations for the functions $N_a(Q,P)$ and for the tensor F^{M}_{ik}, the components of which are the amplitudes of the microscopic electromagnetic field $\boldsymbol{E}^{\mathrm{M}}, \boldsymbol{B}^{\mathrm{M}}$. This set of equations is:

$$u_i \frac{\partial N_a}{\partial Q_i} + \frac{e_a}{c} F^{\mathrm{M}}_{ik} u_k \frac{\partial N_a}{\partial P_i} = 0 \tag{24.16}$$

$$\frac{\partial F^{\mathrm{M}}_{ik}}{\partial Q_k} = 4\pi \sum_a e_a \int d\boldsymbol{p}\, d\mathcal{E}\, u_i N_a(Q,P) \tag{24.17}$$

$$\frac{\partial F^{\mathrm{M}}_{ik}}{\partial Q_l} + \frac{\partial F^{\mathrm{M}}_{kl}}{\partial Q_i} + \frac{\partial F^{\mathrm{M}}_{li}}{\partial Q_k} = 0 \qquad (i,k,l = 1,2,3,4) \tag{24.18}$$

where u_i is the four-velocity vector $u = (\gamma\boldsymbol{v}, ic\gamma)$.

The relativistic invariance of the set of equations (24.16)-(24.18) is manifest. This set can be obtained by means of a variational principle [8].

25. MICROSCOPIC EQUATIONS FOR A COULOMB PLASMA

In the cases where one can limit oneself to the contribution of the Coulomb interactions among the charged particles (Coulomb plasma) Eqs (24.5), (24.6), (24.8)-(24.11) can be considerably simplified and reduce to :

$$\frac{\partial N_a}{\partial t} + \boldsymbol{v}\cdot\frac{\partial N_a}{\partial \boldsymbol{r}} + (e_a \boldsymbol{E}^{\mathrm{M}} + \boldsymbol{F}_{ae})\cdot\frac{\partial N_a}{\partial \boldsymbol{p}} = 0 \tag{25.1}$$

$$\nabla \times \boldsymbol{E}^{\mathrm{M}} = 0, \quad \nabla\cdot\boldsymbol{E}^{\mathrm{M}} = 4\pi \sum_a e_a \int d\boldsymbol{p}\; N_a(\boldsymbol{r},\boldsymbol{p},t). \tag{25.2}$$

In Eq. (25.1) we introduced the external force:

$$\boldsymbol{F}_{a0} = e_a \boldsymbol{E}_0 + \frac{e_a}{c}(\boldsymbol{v}\times\boldsymbol{B}_0) \tag{25.3}$$

$\boldsymbol{E}_0, \boldsymbol{B}_0$ being the amplitudes of the external electric and magnetic fields.

The set (25.1)-(25-2) is a closed set of equations for the phase density N_a and the potential electric field $\boldsymbol{E}^{\mathrm{M}} = -\nabla\phi^{\mathrm{M}}$.

In the Coulomb plasma the state of the system is completely determined by the values of the positions and of the momenta of the particles. Therefore, the field $\boldsymbol{E}^{\mathrm{M}}$ is completely determined by the functions $N_a(x,t)$. Indeed, from Eq. (25.2) follows

$$\begin{aligned}\boldsymbol{E}^{\mathrm{M}}(\boldsymbol{r},t) &= -\nabla\phi^{\mathrm{M}}(\boldsymbol{r},t)\\ &= -\sum_b e_b \frac{\partial}{\partial \boldsymbol{r}} \int dx' \frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|} N_b(x',t).\end{aligned} \tag{25.4}$$

The electric field can be eliminated from (25.1) by means of this equation. As a

result we find a closed set of equations for $N_a(x,t)$:

$$\frac{\partial N_a}{\partial t} + \boldsymbol{v} \cdot \frac{\partial N_a}{\partial \boldsymbol{r}} + \boldsymbol{F}_{a0} \cdot \frac{\partial N_a}{\partial \boldsymbol{p}} - \sum_b \frac{\partial}{\partial \boldsymbol{r}} \cdot \int dx' \, \frac{e_a e_b}{|\boldsymbol{r} - \boldsymbol{r}'|} N_b(x',t) \frac{\partial N_a}{\partial \boldsymbol{p}} = 0 \,. \quad (25.5)$$

This set is analogous to Eq. (4.7) for the phase density of atoms in a gas. For an electron-ion plasma, (25.5) is a set of two equations for the phase densities of electrons, N_e, and of ions, N_i.

Because of their complexity, the microscopic equations cannot be used directly. We therefore go over to the equations for the moments of the random functions $N_a, \boldsymbol{E}^{\mathrm{M}}$.

26. AVERAGING OF THE MICROSCOPIC EQUATIONS

We average Eq. (25.1) and use the definition of the distribution function f_a and of the average field $\boldsymbol{E}$:

$$\left\langle N_a(x,t) \right\rangle = n_a f_a(x,t)$$
$$\left\langle \boldsymbol{E}^{\mathrm{M}} \right\rangle = \boldsymbol{E}(\boldsymbol{r},t) \quad (26.1)$$

and the identity

$$\left\langle \boldsymbol{E}^{\mathrm{M}} N_a(x,t) \right\rangle = \boldsymbol{E}(\boldsymbol{r},t) \, n_a f_a(x,t) + \left\langle \delta E \, \delta N_a \right\rangle_{\boldsymbol{r},x,t} \quad (26.2)$$

As a result we find the equation for the distribution function

$$\frac{\partial f_a}{\partial t} + \boldsymbol{v} \cdot \frac{\partial f_a}{\partial \boldsymbol{r}} + (e_a \boldsymbol{E} + \boldsymbol{F}_{a0}) \cdot \frac{\partial f_a}{\partial \boldsymbol{p}} = -\frac{e_a}{n_a} \frac{\partial}{\partial \boldsymbol{p}} \cdot \left\langle \delta E \;\; \delta N_a \right\rangle_{\boldsymbol{r},x,t} \equiv I_a(x,t) \quad (26.3)$$

$$\nabla \times \boldsymbol{E} = 0 \,, \qquad \nabla \cdot \boldsymbol{E} = 4\pi \sum_a e_a n_a \int d\boldsymbol{p} \, f_a(x,t) \,.$$

Equations (26.3) are analogous to Eq. (6.2) for the gas. By analogy, the right-hand side of (26.3) will be called the collision integral. This name is only given by analogy but does not reflect the essence of the matter, because in a plasma each particle interacts with a large number of other particles. As a result, the model of binary collisions of charged particles (which is the source of the name 'collision integral') is not accurate in a plasma.

For convenience we introduce special notations for the operators $\hat{L}_a, \hat{L}_{ab}, \ldots$; for instance,

$$\hat{L}_{ab} = \left(\frac{\partial}{\partial t} + \boldsymbol{v} \cdot \frac{\partial}{\partial \boldsymbol{r}} + \boldsymbol{v}' \cdot \frac{\partial}{\partial \boldsymbol{r}'} + \boldsymbol{F}_a \cdot \frac{\partial}{\partial \boldsymbol{p}} + \boldsymbol{F}_b \cdot \frac{\partial}{\partial \boldsymbol{p}'} \right) \quad (26.4$$

We denote by $\hat{L}^0_{a0}$ the same operator in which the external force $\boldsymbol{F}_{a0}$ replaces the average force $\boldsymbol{F}_a$. Equations (26.3) thus take the form:

$$\hat{L}_a f_a = -\frac{e_a}{n_a}\frac{\partial}{\partial \boldsymbol{p}}\cdot\langle\delta \boldsymbol{E}\,\delta N_a\rangle_{\boldsymbol{r},x,t} \equiv I_a(x,t)$$

$$\nabla\times\boldsymbol{E} = 0\,, \qquad \nabla\cdot\boldsymbol{E} = 4\pi\sum_b e_b n_b\int d\boldsymbol{p}\, f_b\,. \tag{26.5}$$

These equations are not closed, because of the occurrence of the second moment $\langle\delta \boldsymbol{E}\,\delta N_a\rangle$. Using Eq. (25.4), the latter can be expressed in terms of $\langle\delta N_a\,\delta N_b\rangle$:

$$\langle\delta \boldsymbol{E}\,\delta N_a\rangle_{\boldsymbol{r},x,t} = -\sum_b e_b\int dx'\,\frac{\partial}{\partial \boldsymbol{r}}\frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|}\langle\delta N_a\,\delta N_b\rangle_{x,x',t} \tag{26.6}$$

We now derive an equation for $\langle\delta N_a\,\delta N_b\rangle_{x,x',t}$. We first consider the fluctuations

$$\delta N_a = N_a - \langle N_a\rangle = N_a - n_a f_a\,. \tag{26.7}$$

From (25.1), (25.2), (26.5) we find the equations for the random functions δN_a, $\delta \boldsymbol{E}$:

$$\hat{L}_a\delta N_a + e_a\,\delta\boldsymbol{E}\cdot\frac{\partial\, n_a f_a}{\partial \boldsymbol{p}} = -e_a\frac{\partial}{\partial \boldsymbol{p}}\left\{\delta\boldsymbol{E}\,\delta N_a - \langle\delta\boldsymbol{E}\,\delta N_a\rangle\right\}$$

$$\nabla\times\delta\boldsymbol{E} = 0\,, \qquad \nabla\cdot\delta\boldsymbol{E} = 4\pi\sum_a e_a\int d\boldsymbol{p}\,\delta N_a\,. \tag{26.8}$$

We multiply both sides of (26.8) by $\delta N_b(x',t)$ and average. The corresponding equation for δN_b is multiplied by δN_a and is also averaged. Adding the two equations results in an equation for $\langle\delta N_a\,\delta N_b\rangle$. Using the notation (26.4) we find

$$\begin{aligned}&\hat{L}_{ab}\langle\delta N_a\,\delta N_b\rangle_{x,x',t} + e_a\langle\delta\boldsymbol{E}\,\delta N_b\rangle_{\boldsymbol{r},x,t}\cdot\frac{\partial\, n_a f_a}{\partial \boldsymbol{p}}\\ &\quad + e_b\langle\delta N_a\,\delta\boldsymbol{E}\rangle_{x,\boldsymbol{r}',t}\cdot\frac{\partial\, n_b f_b}{\partial \boldsymbol{p}'}\\ &= -e_a\frac{\partial}{\partial \boldsymbol{p}}\cdot\langle\delta\boldsymbol{E}\,\delta N_a\,\delta N_b\rangle_{\boldsymbol{r},x,x',t} - e_b\frac{\partial}{\partial \boldsymbol{p}'}\cdot\langle\delta N_a\,\delta\boldsymbol{E}\,\delta N_b\rangle_{x,\boldsymbol{r},x',t}\end{aligned} \tag{26.9}$$

This is the second equation in the hierarchy of moment equations for δN_a. It is analogous to (6.6) and is not closed either, because of the occurrence of the third moment.

From the hierarchy of moment equations, one can go over to the hierarchy for the distribution functions $f_a, f_{ab}, f_{abc}, \ldots$ or to a set of equations for the one-particle function f_a and for the correlation functions $g_{ab}, g_{abc}, \ldots$. We now exhibit the formulae relating the second and third moments to the functions f_a, g_{ab}, g_{abc}. These formulae are analogous to (5.6), (5.10):

$$\left\langle \delta N_a\, \delta N_b \right\rangle_{x,x',t} = \frac{N_a N_b - \delta_{ab} N_a}{V^2}\, g_{ab}(x,x',t) + \frac{N_a}{V}\,\delta_{ab}\left[\delta(x-x')\, f_a(x,t) - \frac{1}{V} f_a(x,t)\, f_b(x',t)\right] \tag{26.10}$$

Hence, in the limit $N \to \infty$, $V \to \infty$, $N/V = n = \text{const}$:

$$\left\langle \delta N_a\, \delta N_b \right\rangle_{x,x',t} = n_a n_b g_{ab}(x,x',x) + n_a \delta_{ab}\delta(x-x')\, f_a(x,t) \tag{26.11}$$

In the same limit, and for $n_a = n_b = n$:

$$\begin{aligned}\left\langle \delta N_a\, \delta N_b\, \delta N_c \right\rangle_{x,x',x'',t} &= n^3 g_{abc}(x,x',x'',t) \\ &+ n^2\Big[\delta_{ab}\,\delta(x-x')\, g_{bc}(x',x'',t) + \delta_{bc}\,\delta(x'-x'')\, g_{ac}(x,x'',t) \\ &+ \delta_{ac}\,\delta(x-x'')\, g_{ab}(x,x',t)\Big] + n\,\delta_{ab}\,\delta_{bc}\,\delta(x-x')\,\delta(x'-x'')\, f_c(x'',t)\end{aligned} \tag{26.12}$$

These equations will be used in the next section.

In the general case, when the full electromagnetic field is considered, one must start from Eqs (24.5), (24.6), (24.8)-(24.11) for the random functions $N_a, \boldsymbol{E}^{\mathrm{M}}, \boldsymbol{B}^{\mathrm{M}}$. Averaging these equations and using (26.1), (26.2) and the corresponding equations for the magnetic field, we find

$$\frac{\partial f_a}{\partial t} + \boldsymbol{v}\cdot\frac{\partial f_a}{\partial \boldsymbol{r}} + \boldsymbol{F}_a\cdot\frac{\partial f_a}{\partial \boldsymbol{p}} = -\frac{1}{n_a}\frac{\partial}{\partial \boldsymbol{p}}\cdot\left\langle \delta\boldsymbol{F}_a\, \delta N_a\right\rangle_{x,x',t} \equiv I_a(x,t) \tag{26.13}$$

$$\boldsymbol{F}_a = \boldsymbol{F}_{a0} + e_a \boldsymbol{E} + \frac{e_a}{c}(\boldsymbol{v}\times\boldsymbol{B}) \tag{26.14}$$

$$\delta\boldsymbol{F}_a = e_a\,\delta\boldsymbol{E} + \frac{e_a}{c}(\boldsymbol{v}\times\delta\boldsymbol{B}) \tag{26.15}$$

$$\nabla\times\boldsymbol{B} = \frac{1}{c}\frac{\partial \boldsymbol{E}}{\partial t} + \frac{4\pi}{c}\sum_a e_a n_a \int d\boldsymbol{p}\,\boldsymbol{v}\, f_a \tag{26.16}$$

$$\nabla\times\boldsymbol{E} = -\frac{1}{c}\frac{\partial \boldsymbol{B}}{\partial t}, \qquad \nabla\cdot\boldsymbol{B} = 0 \tag{26.17}$$

$$\nabla\cdot\boldsymbol{E} = 4\pi\sum_a e_a n_a \int d\boldsymbol{p}\, f_a\,. \tag{26.18}$$

As before, these equations for $f_a, \boldsymbol{E}, \boldsymbol{B}$ are not closed, because of the occurrence of the second moment $\left\langle \delta\boldsymbol{F}_a\, \delta N_a\right\rangle$. The latter is defined in terms of the correlations of the fluctuations δN_a, $\delta\boldsymbol{E}$, $\delta\boldsymbol{B}$.

We now write the equations for δN_a, $\delta\boldsymbol{E}$, $\delta\boldsymbol{B}$. They follow from (24.5), (24.6), (24.8)-(24.11), (26.13)-(26.18):

$$\hat{L}_a \delta N_a + \delta \boldsymbol{F}_a \cdot \frac{\partial n_a f_a}{\partial \boldsymbol{p}} = -\frac{\partial}{\partial \boldsymbol{p}} \cdot \left\{ \delta \boldsymbol{F}_a \delta N_a - \langle \delta \boldsymbol{F}_a \delta N_a \rangle \right\} \tag{26.19}$$

$$\nabla \times \delta \boldsymbol{B} = \frac{1}{c} \frac{\partial \boldsymbol{E}}{\partial t} + \frac{4\pi}{c} \sum_a e_a \int d\boldsymbol{p}\, \boldsymbol{v}\, \delta N_a \tag{26.20}$$

$$\nabla \times \delta \boldsymbol{E} = -\frac{1}{c} \frac{\partial \delta \boldsymbol{B}}{\partial t} \tag{26.21}$$

$$\nabla \cdot \delta \boldsymbol{B} = 0 \tag{26.22}$$

$$\nabla \cdot \delta \boldsymbol{E} = 4\pi \sum_a e_a \int d\boldsymbol{p}\, \delta N_a \,. \tag{26.23}$$

From these equations one can derive an equation for the second moments of the fluctuations $\delta N_a, \delta \boldsymbol{E}, \delta \boldsymbol{B}$ which, of course, will contain the third moments. We thus obtain an infinite hierarchy of equations for the moments of the phase densities and for the electromagnetic field amplitudes.

We now show that, to first order in the plasma parameter (in the polarization approximation), this hierarchy is limited.

27 APPROXIMATION OF BINARY COLLISIONS AND POLARIZATION APPROXIMATION FOR PLASMAS

The equation for $\langle \delta N_a \delta N_b \rangle_{x,x',t}$ in the polarization approximation is analogous to (6.13):

$$\hat{L}_{ab} \langle \delta N_a \delta N_b \rangle_{x,x',t} + \langle \delta \boldsymbol{F}_a \delta N_b \rangle_{x,x',t} \cdot \frac{\partial n_a f_a}{\partial \boldsymbol{p}}$$

$$+ \langle \delta N_a \delta \boldsymbol{F}_b \rangle_{x,x',t} \cdot \frac{\partial n_b f_b}{\partial \boldsymbol{p}'} = \hat{L}_{ab} \langle \delta N_a \delta N_b \rangle^{\text{source}}_{x,x',t} \tag{27.1}$$

$$\langle \delta N_a \delta N_b \rangle^{\text{source}}_{x,x',t} = n_a \delta_{ab} \left[\delta(x-x') f_a(x,t) - \frac{1}{V} f_a(x,t) f_b(x',t) \right] . \tag{27.2}$$

In the limit $N \to \infty$, $V \to \infty$, $N/V = n$, Eq. (27.2) becomes

$$\langle \delta N_a \delta N_b \rangle^{\text{source}}_{x,x',t} = n_a \delta_{ab} \delta(x-x') f_a(x,t) \tag{27.3}$$

We recall that the superscript 'source' denotes the fact that the right-hand side of (27.1) is independent of the second moment, and is completely defined by the one-particle distribution function f_a.

Using (26.10) and (26.13), we may write, in the same approximation, the equation for the correlation function g_{ab}:

$$\hat{L}_{ab} g_{ab}(x,x',t) = \hat{\theta}_{ab} f_a(x,t)\, f_b(x',t)$$
$$+ \sum_c n_c \int dx'' \left[\hat{\theta}_{ac} g_{cb}(x'',x',t)\, f_a(x,t) + \hat{\theta}_{bc}\, g_{ac}(x,x'',t)\, f_b(x',t)\right] \qquad (27.4)$$

where we introduce the operator [see (1.6)]:

$$\hat{\theta}_{ab} = \frac{\partial \Phi_{ab}}{\partial \boldsymbol{r}} \cdot \frac{\partial}{\partial \boldsymbol{p}} + \frac{\partial \Phi_{ab}}{\partial \boldsymbol{r}'} \cdot \frac{\partial}{\partial \boldsymbol{p}'} \qquad (27.5)$$

with

$$\Phi_{ab} = \frac{e_a e_b}{|\boldsymbol{r} - \boldsymbol{r}'|} .$$

Equation (27.4) is analogous to (3.5). The second term on its right-hand side, proportional to n_c, accounts for the polarization effects, as will be seen later. In perturbation theory this term can, to first order in ξ, be neglected. As a result, the equation for g_{ab} to this order reduces to

$$\hat{L}_{ab}\, g_{ab}\,(x,x',t) = \hat{\theta}_{ab}\, f_a(x,t)\, f_b\,(x',t) \qquad (27.6)$$

and corresponds to Eq. (3.4).

We shall not write here the equations for the two-time, one-particle function $F_a\,(x,t,x',t')$ and for the two-time moment $\langle \delta N_a\, \delta N_b \rangle_{x,t,x',t'}$: they correspond to Eqs (6.24), (6.17). Thus, just as for the gases, the difference between the polarization approximation and the second moment approximation is the inclusion of source terms defined by the one-particle functions f_a, F_a.

On that basis, just as in section 6, we introduce a corresponding source term in the equation for the fluctuations δN_a. In the second moment approximation, the right-hand side of (26.19) is set equal to zero. In the polarization approximation, there appears a source term corresponding to the source term in the equations for $\langle \delta N_a\, \delta N_b \rangle_{x,x',t}$ and $\langle \delta N_a\, \delta N_b \rangle_{x,x',t,t'}$

$$\hat{L}_{ab}\, \delta N_a + \delta \boldsymbol{F}_a \cdot \frac{\partial\, n_a f_a}{\partial \boldsymbol{p}} = \hat{L}_a\; \delta N_a^{\mathbf{source}} . \qquad (27.7)$$

The correlation of the source fluctuations is defined by the equation:

$$\hat{L}_a \langle \delta N_a\, \delta N_b \rangle^{\mathbf{source}}_{x,t,x',t'} = 0 \qquad (27.8)$$

with the initial condition

$$\langle \delta N_a\, \delta N_b \rangle^{\mathbf{source}}_{x,t,x',t'} \Big|_{t'=t}$$
$$= n_a\, \delta_{ab} \left[\delta(x - x')\; f_a(x,t) - \frac{1}{V}\, f_a(x,t)\, f_b\,(x',t)\right] \qquad (27.9)$$

which in the limit $N \to \infty$, $V \to \infty$, $N/V = n$. reduces to

$$\left\langle \delta N_a \, \delta N_b \right\rangle^{\text{source}}_{x,t,x',t'} \Big|_{t'=t} = n_a \, \delta_{ab} \, \delta(x - x') f_a(x,t) \tag{27.10}$$

In the general case, Eq. (27.7) must be completed with Eqs (26.20)-(26.23) for the fluctuations of the electromagnetic field amplitudes and the relation

$$\delta \boldsymbol{F}_a = e_a \, \delta \boldsymbol{E} + \frac{e_a}{c} (\boldsymbol{v} \times \delta \boldsymbol{B}) . \tag{27.11}$$

In the case of a Coulomb plasma, we only need the equations

$$\nabla \times \delta \boldsymbol{E} = 0$$

$$\nabla \cdot \delta \boldsymbol{E} = 4\pi \sum_a e_a \int d\boldsymbol{p} \; \delta N_a \tag{27.12}$$

together with

$$\delta \boldsymbol{F}_a = e_a \, \delta \boldsymbol{E} . \tag{27.13}$$

The results obtained here will be used in the derivation of kinetic equations.

28 TRANSPORT EQUATIONS FOR THE DENSITY, THE MOMENTUM DENSITY AND THE ENERGY DENSITY OF THE PARTICLES. TRANSPORT EQUATIONS FOR THE ENERGY DENSITY AND THE MOMENTUM DENSITY OF THE ELECTROMAGNETIC FIELD

For the derivation of the transport equations we use the set of equations (26.13)-(26.18) for f_a, $\boldsymbol{E}$, $\boldsymbol{B}$.

The total mass density, the total momentum density and the total kinetic energy density for the particles of all kinds are defined by analogy to (7.1):

$$\rho(\boldsymbol{r},t) = \sum_a m_a n_a \int d\boldsymbol{p} \; f_a$$

$$\sum_a \rho_a \boldsymbol{u}_a = \sum_a n_a \int d\boldsymbol{p} \, \boldsymbol{p} \, f_a$$

$$W(\boldsymbol{r},t) = \sum_a n_a \int d\boldsymbol{p} \, \frac{p^2}{2m_a} f_a \tag{28.1}$$

Multiplying Eq. (26.13) by $n_a m_a$, integrating over $\boldsymbol{p}$ and summing over a, we find the continuity equation for the total density of the plasma:

$$\frac{\partial \rho(\boldsymbol{r},t)}{\partial t} + \nabla \cdot \sum_a \rho_a \boldsymbol{u}_a = 0 . \tag{28.2}$$

We consider now the momentum balance of the plasma. The momentum flux tensor Π_{ij} is defined as follows

$$\Pi_{ij} = \sum_a m_a n_a \int d\boldsymbol{p} \, v_i v_j f_a = \sum_a n_a m_a u_{ai} u_{aj} + P_{ij} , \tag{28.3}$$

where P_{ij} is the pressure tensor

$$P_{ij} = \sum_a m_a n_a \int d\boldsymbol{p}\, \delta v_{ai}\, \delta v_{aj} f_a \equiv \sum_a P_{ij}^{(a)} \tag{28.4}$$

with $\delta v_{ai} = v_i - u_{ai}$; $P_{ij}^{(a)}$ is the pressure tensor of component a. Just as in the case of gases, the pressure tensor can be represented as follows

$$\begin{aligned} P_{ij}^{(a)} &= \delta_{ij}\, p^{(a)} + \pi_{ij}^{(a)} \\ p^{(a)} &= \tfrac{1}{3} P_{ii}^{(a)} \end{aligned} \tag{28.5}$$

where $p^{(a)}$ is the scalar (hydrostatic) pressure of component a and $\pi_{ij}^{(a)}$ the corresponding tensor of viscous stresses.

We finally introduce the definitions of the charge and current densities,

$$\begin{aligned} q &= \sum_a e_a n_a \int d\boldsymbol{p}\, f_a \\ \boldsymbol{j} &= \sum_a e_a n_a \int d\boldsymbol{p}\, \boldsymbol{v}\, f_a \, . \end{aligned} \tag{28.6}$$

We now multiply (26.13) by np_i, integrate over $\boldsymbol{p}$, sum over a, and perform the transformations leading to Eq. (7.9). In the result, we obtain the balance equation for the total momentum density of the particles, analogous to (7.9):

$$\begin{aligned} &\frac{\partial}{\partial t} \sum_a \rho_a u_{ai} + \frac{\partial}{\partial r_j} \sum_a \left(\rho_a u_{ai} u_{aj} + \delta_{ij}\, p^{(a)} + \pi_{ij}^{(a)} \right) \\ &\qquad = qE_i + \frac{1}{c} (\boldsymbol{j} \times \boldsymbol{B})_i + \sum_a n_a \int d\boldsymbol{p}\; p_i I_a \; . \end{aligned} \tag{28.7}$$

The first and second term on the right-hand side represent the i$^{\text{th}}$ component of the force density: in order to derive this expression we used Eq. (26.14). The third term on the right-hand side defines the contribution of the collision integral to the momentum balance. It can be written in a more convenient form. We use the relation of the collision integral I_a to the correlation of the force fluctuations and δN_a; as well as Eq. (26.15) and the definitions

$$\begin{aligned} \delta q &= \sum_a e_a \int d\boldsymbol{p}\; \delta N_a \\ \delta \boldsymbol{j} &= \sum_a e_a \int d\boldsymbol{p}\, \boldsymbol{v}\; \delta N_a \; . \end{aligned}$$

We then find

$$\sum_a n_a \int d\boldsymbol{p}\, \boldsymbol{p}\, I_a = \langle \delta q\; \delta \boldsymbol{E} \rangle + \frac{1}{c} \langle \delta \boldsymbol{j} \times \delta \boldsymbol{B} \rangle \tag{28.8}$$

We now consider the kinetic energy density balance. We multiply Eq. (26.13) by $n_a p^2/2m_a$, integrate over $\boldsymbol{p}$, sum over a and perform the transformations leading to (7.14). As a result we find :

$$\frac{\partial}{\partial t}\sum_a\left(\frac{\rho_a u_a^2}{2}+\frac{3}{2}\frac{\rho_a}{m_a}k_B T_a\right)$$

$$+\frac{\partial}{\partial r_j}\sum_a\left\{u_{ai}\left(\frac{\rho_a u_a^2}{2}+\frac{3\rho_a}{2m_a}k_B T_a+p^{(a)}\right)+\pi_{ij}^{(a)}u_{aj}+S_{ai}\right\}$$

$$=\boldsymbol{j}\cdot\boldsymbol{E}+\sum_a n_a\int d\boldsymbol{p}\,\frac{p^2}{2m_a}I_a \tag{28.9}$$

where T_a is the temperature of component a, $\boldsymbol{S}_a$ the heat flow vector of component a.

Using the definition (26.13) of the collision integral, the second term on the right-hand side can be rewritten as

$$\sum_a n_a\int d\boldsymbol{p}\,\frac{p^2}{2m_a}I_a=\langle\delta\boldsymbol{j}\cdot\delta\boldsymbol{E}\rangle. \tag{28.10}$$

From Eqs (28.8), (28.10) one sees that the contributions of the collision integral to the momentum and energy balance of the plasma are expressible in terms of the correlations of δq, $\delta\boldsymbol{j}$ and $\delta\boldsymbol{E}$, $\delta\boldsymbol{B}$.

We now consider the equation for the momentum balance of the electromagnetic field. To this purpose, by using Maxwell's equations (26.16)-(26.18), we transform the expressions of the electromagnetic forces appearing on the right-hand side of (28.7) :

$$q\boldsymbol{E}+\frac{1}{c}(\boldsymbol{j}\times\boldsymbol{B})$$

$$=\frac{1}{4\pi}(\nabla\cdot\boldsymbol{E})\boldsymbol{E}+\frac{1}{4\pi}(\nabla\times\boldsymbol{B})\times\boldsymbol{B}-\frac{1}{4\pi c}\left(\frac{\partial\boldsymbol{E}}{\partial t}\right)\times\boldsymbol{B}.$$

Hence, from (26.17)

$$q\boldsymbol{E}+\frac{1}{c}(\boldsymbol{j}\times\boldsymbol{B})=\frac{1}{4\pi}\left[\boldsymbol{E}(\nabla\cdot\boldsymbol{E})+\boldsymbol{B}(\nabla\cdot\boldsymbol{B})\right]$$

$$+\frac{1}{4\pi}\left\{(\nabla\times\boldsymbol{B})\times\boldsymbol{B}+(\nabla\times\boldsymbol{E})\times\boldsymbol{E}\right\}-\frac{1}{4\pi c}\frac{\partial}{\partial t}(\boldsymbol{E}\times\boldsymbol{B}). \tag{28.11}$$

We now use the vector identity

$$A_x(\nabla\cdot\boldsymbol{A})+\left[(\nabla\times\boldsymbol{A})\times\boldsymbol{A}\right]_x$$

$$=\frac{\partial}{\partial x}\left(A_x^2-\frac{1}{2}A^2\right)+\frac{\partial}{\partial y}(A_x A_y)+\frac{\partial}{\partial z}(A_x A_z) \tag{28.12}$$

Substituting successively $A_i=E_i$ and $A_i=B_i$, and adding, we transform

Eq. (28.11) into

$$\frac{\partial}{\partial t} \frac{(\boldsymbol{E} \times \boldsymbol{B})_i}{4\pi c} + \frac{\partial}{\partial r_j} T_{ij} = -qE_i - \frac{1}{c} (\boldsymbol{j} \times \boldsymbol{B})_i \tag{28.13}$$

The tensor T_{ij} is called the electromagnetic pressure tensor or the Maxwell tensor and is defined as follows

$$T_{ij} = -\frac{1}{4\pi} \left(E_i E_j + B_i B_j - \delta_{ij} \frac{E^2 + B^2}{2} \right). \tag{28.14}$$

From (28.13) follows that the vector

$$\frac{1}{4\pi c} (\boldsymbol{E} \times \boldsymbol{B}) \tag{28.15}$$

defines the momentum density of the average electromagnetic field.

Adding term by term the particle momentum balance (28.7) and the average field momentum balance (28.13) we obtain the total momentum balance for the plasma and the average field:

$$\frac{\partial}{\partial t} \left\{ \sum_a \rho_a u_{ia} + \frac{1}{4\pi c} (\boldsymbol{E} \times \boldsymbol{B})_i \right\}$$
$$+ \frac{\partial}{\partial r_j} \left\{ \sum_a \left(\rho_a u_{ai} u_{aj} + P_{ij}^{(a)} \right) + T_{ij} \right\} = \sum_a n_a \int d\boldsymbol{p}\, p_i I_a \tag{28.16}$$

The right-hand side of this equation can be transformed by using Eqs (28.8) and (26.20)-(26.23). Performing a transformation analogous to the one leading to Eq. (28.13), we obtain instead of (28.8)

$$\sum_a n_a \int d\boldsymbol{p}\, p_i I_a = - \frac{\partial}{\partial t} \frac{\langle \delta \boldsymbol{E}\ \delta \boldsymbol{B} \rangle_i}{4\pi c} \frac{\partial}{\partial r_j} K_{ij} \tag{28.17}$$

where we introduce the tensor K_{ij} of the electromagnetic pressure due to the field fluctuations:

$$K_{ij} = -\frac{1}{4\pi} \left\{ \langle \delta E_i \delta E_j \rangle + \langle \delta B_i\ \delta B_j \rangle - \delta_{ij} \frac{\langle \delta E^2 \rangle + \langle \delta B^2 \rangle}{2} \right\} \tag{28.18}$$

Combining (28.16) and (28.17) we find

$$\frac{\partial}{\partial t} \left\{ \sum_a \rho_a u_{ai} + \frac{1}{4\pi c} [(\boldsymbol{E} \times \boldsymbol{B})_i + \langle \delta \boldsymbol{E} \times \delta \boldsymbol{B} \rangle_i] \right\}$$
$$+ \frac{\partial}{\partial r_j} \left\{ \sum_a \left[\rho_a u_{ai} u_{aj} + P_{ij}^{(a)} \right] + T_{ij} + K_{ij} \right\} = 0 \tag{28.19}$$

It thus follows that the total momentum density of the particles and of the fields, including the field fluctuations, is defined by the vector $\boldsymbol{G}$:

$$\boldsymbol{G} = \sum_a \rho_a \boldsymbol{u}_a + \frac{1}{4\pi c} \left\{ (\boldsymbol{E} \times \boldsymbol{B}) + \langle \delta \boldsymbol{E} \times \delta \boldsymbol{B} \rangle \right\}$$
$$\equiv \sum_a \rho_a \boldsymbol{u}_a + \frac{1}{4\pi c} \langle \boldsymbol{E}^{\mathbf{M}} \times \boldsymbol{B}^{\mathbf{M}} \rangle . \tag{28.20}$$

The expression appearing in (28.19) under the action of $\partial/\partial r_j$ defines the total momentum flux tensor of particles and field.

We finally consider the electromagnetic energy balance. Multiply (26.16) scalarly by $\boldsymbol{E}$ and (26.17) by $\boldsymbol{B}$, subtract the second from the first and use the identity:

$$\nabla\cdot(\boldsymbol{E}\times\boldsymbol{B}) = \boldsymbol{B}\cdot(\nabla\times\boldsymbol{E}) - \boldsymbol{E}\cdot(\nabla\times\boldsymbol{B})\ . \tag{28.21}$$

We thus obtain the energy density balance for the average field

$$\frac{\partial}{\partial t}\frac{E^2+B^2}{8\pi} + \nabla\cdot\frac{c}{4\pi}(\boldsymbol{E}\times\boldsymbol{B}) = -\boldsymbol{j}\cdot\boldsymbol{E}\ . \tag{28.22}$$

The vector

$$\boldsymbol{S}_F = \frac{c}{4\pi}(\boldsymbol{E}\times\boldsymbol{B}) \tag{28.23}$$

is the energy flux vector of the average field, i.e. the Poynting vector. We now add Eqs (28.9), (28.22):

$$\begin{aligned}\frac{\partial}{\partial t}\Bigg\{\sum_a\left(\frac{\rho_a u_a^2}{2} + \frac{3\rho_a}{2m_a}k_B T_a\right) + \frac{E^2+B^2}{8\pi}\Bigg\}\\ + \frac{\partial}{\partial r_j}\Bigg\{\sum_a\left[u_{aj}\left(\frac{\rho_a u_a^2}{2} + \frac{3\rho_a}{2m_a}k_B T_a + p^{(a)}\right) + \pi_{jn}^{(a)}u_{an} + S_j^{(a)}\right]\\ + \frac{c}{4\pi}(\boldsymbol{E}\times\boldsymbol{B})_j\Bigg\} = \sum_a n_a\int d\boldsymbol{p}\,\frac{p^2}{2m_a}I_a\ .\end{aligned} \tag{28.24}$$

On the left-hand side there appear the total energy density and total energy flux density of the particles and of the average field, without accounting for the fluctuations. The contribution of the latter is contained on the right-hand side, through the collision integral, just as in the case of the momentum balance.

We transform the right-hand side of (28.24) by using (28.10), (26.20) and (26.21). Proceeding as for the derivation of (28.22), we obtain:

$$\sum_a n_a\int d\boldsymbol{p}\,\frac{p^2}{2m_a}I_a = \langle\delta\boldsymbol{j}\cdot\delta\boldsymbol{E}\rangle = -\frac{\partial}{\partial t}\frac{\langle\delta E^2\rangle + \langle\delta B^2\rangle}{8\pi} - \nabla\cdot\frac{c}{4\pi}\langle\delta\boldsymbol{E}\times\delta\boldsymbol{B}\rangle\ . \tag{28.25}$$

Substituting into (28.24) we find the exact balance equation for the total energy density of particles and field, including the effect of fluctuations:

$$\begin{aligned}\frac{\partial}{\partial t}\Bigg\{\sum_a\left[\frac{\rho_a u_a^2}{2} + \frac{3\rho_a}{2m_a}k_B T_a\right] + \frac{1}{8\pi}(E^2+B^2) + \frac{1}{8\pi}\Big(\langle\delta E^2\rangle + \langle\delta B^2\rangle\Big)\Bigg\}\\ + \frac{\partial}{\partial r_i}\Bigg\{\sum_a\left[u_{ai}\left(\frac{\rho_a u_a^2}{2} + \frac{3\rho_a}{2m_a}k_B T_a + p^{(a)}\right) + \pi_{ij}^{(a)}u_{aj} + S_{ai}\right]\\ + \frac{c}{4\pi}(\boldsymbol{E}\times\boldsymbol{B})_i + \frac{c}{4\pi}\langle\delta\boldsymbol{E}\times\delta\boldsymbol{B}\rangle_i\Bigg\} = 0\ .\end{aligned} \tag{28.26}$$

For the case of a Coulomb plasma, the balance equations are simplified: $\delta\boldsymbol{B} = 0$ and $\boldsymbol{B}$ reduces to the external magnetic field, $\boldsymbol{B}_0$.

CHAPTER 6

Kinetic Equations for the Plasma in the First Moment Approximation. The Vlasov Equation

29. KINETIC DESCRIPTION OF THE PROCESSES IN A COLLISIONLESS PLASMA

We introduce the well-known relaxation time of a plasma, $\tau_{\mathbf{rel}}$, defined as the time necessary for reaching local equilibrium; $l_{\mathbf{rel}}$ is the corresponding length. In the case of ordinary gases we had called the corresponding quantities: $\tau_{\mathbf{rel}}$ (the average time between two successive collisions) and l (the mean free path of the atoms). Sometimes, we shall use this terminology also for plasmas.

The plasma consists of several components. In the simplest case, there are two components. It follows that the relaxation processes in a plasma proceed in several steps, characterized by different relaxation times, $\tau_{\mathbf{ab}}$. For an electron-ion plasma, $a = e, i$, hence there are four of these: τ_{ee}, τ_{ei}, τ_{ie}, τ_{ii}.

The relaxation processes are caused by the existence of fluctuations. Their contribution to the kinetic equation for the functions f_a is determined by the collision integrals:

$$\begin{aligned} I_a &= -\frac{1}{n_a}\frac{\partial}{\partial \boldsymbol{p}} \cdot \left\langle \delta \boldsymbol{F}_a \, \delta N_a \right\rangle \\ &= -\frac{e_a}{n_a}\frac{\partial}{\partial \boldsymbol{p}} \cdot \left\{ \left\langle \delta \boldsymbol{E} \;\; \delta N_a \right\rangle + \frac{1}{c}\,\boldsymbol{v} \times \left\langle \delta \boldsymbol{B} \;\; \delta N_a \right\rangle \right\} \end{aligned} \tag{29.1}$$

We used here the definitions (26.13) and (26.14).

The kinetic equations for plasmas are closed equations for the one-particle distribution functions f_a and for the average fields, $\boldsymbol{E}, \boldsymbol{B}$; the collision integrals therefore depend on these quantities

$$I_a = I_a(f_a, \boldsymbol{E}, \boldsymbol{B}) \; . \tag{29.2}$$

If the fields are not very strong (see sect. 37), the collision integrals are practically independent of them:

$$I_a = I_a(f_a) \; . \tag{29.3}$$

The relaxation times are defined in such a way that

$$I_a \sim \frac{1}{\tau_a} f_a\,, \quad \frac{1}{\tau_a} = \sum_b \frac{1}{\tau_{ab}}\,. \tag{29.4}$$

In section 38 it will be shown that the shortest relaxation time $\tau_{\mathbf{rel}}$ and the corresponding shortest relaxation length $l_{\mathbf{rel}}$ are defined in terms of the quantities r_{D}, μ and v_t as follows :

$$l_{\mathbf{rel}} \sim r_{\mathrm{D}}/\mu\,, \qquad \tau_{\mathbf{rel}} = l_{\mathbf{rel}}/v_T \sim r_{\mathrm{D}}/v_T\,\mu\,. \tag{29.5}$$

Therefore, for a dilute plasma, for which $\mu \ll 1$, we have

$$l_{\mathbf{rel}} \gg r_{\mathrm{D}}\,, \qquad \tau_{\mathbf{rel}} \gg r_{\mathrm{D}}/v_T\,. \tag{29.6}$$

The relaxation length is much longer than the quantity r_{D}, which defines the correlation radius of the density fluctuations, as well as the effective interaction range of the charged particles.

If the characteristic dimension of the volume enclosing the plasma, L, and the corresponding time T are such that

$$l_{\mathbf{rel}} \gg L \gg r_{\mathrm{D}}\,, \qquad \tau_{\mathbf{rel}} \gg T \gg r_{\mathrm{D}}/v_T\,,$$

then, to zeroth order in $L/l_{\mathbf{rel}}$, $T/\tau_{\mathbf{rel}}$, the term of Eq. (26.13) containing the fluctuations $\delta \boldsymbol{F}_a$, δN_a can be neglected. In other words, the contribution of the second moments can be disregarded. As a result, we obtain a closed set of equations for the first moments of the random functions N_a, $\boldsymbol{E}^{\mathrm{M}}$, $\boldsymbol{B}^{\mathrm{M}}$, i.e. for the functions $f_a = \langle N_a \rangle / n_a$, $\boldsymbol{E}$, $\boldsymbol{B}$:

$$\frac{\partial f_a}{\partial t} + \boldsymbol{v} \cdot \frac{\partial f_a}{\partial \boldsymbol{r}} + \boldsymbol{F}_a \cdot \frac{\partial f_a}{\partial \boldsymbol{p}} = 0 \tag{29.7}$$

$$\boldsymbol{F}_a = \boldsymbol{F}_{a0} + e_a \boldsymbol{E} + \frac{e_a}{c}\,(\boldsymbol{v} \times \boldsymbol{B}) \tag{29.8}$$

$$\nabla \times \boldsymbol{B} = \frac{1}{c}\,\frac{\partial \boldsymbol{E}}{\partial t} + \frac{4\pi}{c} \sum_a e_a n_a \int d\boldsymbol{p}\; \boldsymbol{v}\, f_a \tag{29.9}$$

$$\nabla \times \boldsymbol{E} = -\frac{1}{c}\,\frac{\partial \boldsymbol{B}}{\partial t} \tag{29.10}$$

$$\nabla \cdot \boldsymbol{B} = 0 \tag{29.11}$$

$$\nabla \cdot \boldsymbol{E} = 4\pi \sum_a e_a n_a \int d\boldsymbol{p}\; f_a\,. \tag{29.12}$$

This self-consistent set of equations was first studied by A.A. Vlasov and was given his name.

For Coulomb plasmas, Eqs (29.8) - (29.12) simplify to :

$$\boldsymbol{F}_a = \boldsymbol{F}_{a0} + e_a \boldsymbol{E} \tag{29.13}$$

$$\nabla \times \boldsymbol{E} = 0\,, \qquad \nabla \cdot \boldsymbol{E} = 4\pi \sum_a e_a n_a \int d\boldsymbol{p}\, f_a\,. \tag{29.14}$$

Instead of the Poisson equation for $\boldsymbol{E}$ it is sometimes convenient to use the equation for the total current. Equations (29.14) are then changed into

$$\nabla \times \boldsymbol{E} = 0\,, \qquad \frac{\partial \boldsymbol{E}}{\partial t} + 4\pi \sum_a e_a n_a \int d\boldsymbol{p}\, \boldsymbol{v}\, f_a = 0\,. \tag{29.15}$$

Equations (29.7) - (29.12) are also valid for relativistic plasmas, provided the relation $\boldsymbol{p} = m_a \boldsymbol{v}$ is changed into the relativistic relation $\boldsymbol{p} = m_a \gamma \boldsymbol{v}$.

In the first moment approximation, the correlation effects are not included; therefore these equations cannot describe dissipative processes. The total entropy of the plasma:

$$S(t) = -k_B \sum_a n_a \int d\boldsymbol{r}\, d\boldsymbol{p}\, f_a \,\ln f_a \tag{29.16}$$

is therefore constant:

$$dS/dt = 0\,. \tag{29.17}$$

In order to prove this result, it is sufficient to multiply Eq. (29.7) by $-k_B n_a \ln f_a$, integrate over $\boldsymbol{r}$ and $\boldsymbol{p}$ and sum over a.

In the language of gas theory, Eqs. (29.7) - (29.12) correspond to the complete neglect of the collisions of the charged particles. They are said to describe processes in a 'collisionless plasma'.

For a collisionless plasma we may use the balance equations for the mass, momentum and energy densities, obtained in section 28, provided we neglect all terms involving the integrals I_a and the correlations of the fluctuations δq, $\delta \boldsymbol{j}$, $\delta \boldsymbol{E}$ and $\delta \boldsymbol{B}$.

Because of the neglect of dissipative processes, Eqs (29.7) - (29.12) cannot be used to describe the approach to equilibrium. As a result, in the absence of an external field ($\boldsymbol{E}_{a0} = 0$), the solution of Eqs. (29.7) - (29.12) with $\boldsymbol{E} = 0$, $\boldsymbol{B} = 0$ is any arbitrary function of the absolute value of the momentum, $f_a(|\boldsymbol{p}|)$.

30. THE LINEAR APPROXIMATION

The set of self-consistent equations, in the absence of external fields ($\boldsymbol{F}_{a0} = 0$) has a particular solution:

$$f_a = f_a^0(|\mathbf{p}|)\,, \quad \boldsymbol{E} = 0\,, \quad \boldsymbol{B} = 0\,. \tag{30.1}$$

Indeed

$$q = \sum_a e_a n_a \int d\boldsymbol{p}\, f_a^0(\mathrm{p}) = \sum_a e_a n_a = 0 \tag{30.2}$$

as a result of the overall neutrality of the plasma, and

$$j = \sum_a e_a n_a \int d\boldsymbol{p}\, \boldsymbol{v}\, f_a^0(p) = 0 \ . \tag{30.3}$$

Thus, the equations for the fields are homogeneous and have the solution $\boldsymbol{E}=0$, $\boldsymbol{B}=0$.

We now consider a solution of (29.7) - (29.12), close to the particular solution (30.1). Thus,

$$f_a = f_a^0 + f_a^1(\boldsymbol{r},\boldsymbol{p},t) \ , \qquad f_a^1 \ll f_a^0$$
$$\boldsymbol{E} = \boldsymbol{E}^1 , \qquad \boldsymbol{B} = \boldsymbol{B}^1 \tag{30.4}$$

From (29.7) - (29.12) we get the following equations, linearized in f_a^1, $\boldsymbol{E}^1$, $\boldsymbol{B}^1$:

$$\frac{\partial f_a^1}{\partial t} + \boldsymbol{v}\cdot\frac{\partial f_a^1}{\partial \boldsymbol{r}} + e_a \boldsymbol{E}^1 \cdot \frac{\partial f_a^0}{\partial \boldsymbol{p}} = 0 \ , \tag{30.5}$$

$$\nabla\times\boldsymbol{B}^1 = \frac{1}{c}\frac{\partial \boldsymbol{E}^1}{\partial t} + \frac{4\pi}{c}\sum_a e_a n_a \int d\boldsymbol{p}\, \boldsymbol{v}\, f_a^1 , \tag{30.6}$$

$$\nabla\times\boldsymbol{E}^1 = -\frac{1}{c}\frac{\partial \boldsymbol{B}^1}{\partial t} \tag{30.7}$$

$$\nabla\cdot\boldsymbol{B}^1 = 0 \tag{30.8}$$

$$\nabla\cdot\boldsymbol{E}^1 = 4\pi\sum_a e_a n_a \int d\boldsymbol{p}\, f_a^1 \ . \tag{30.9}$$

We used the fact that the term involving $\boldsymbol{B}^1$ in (30.5) vanishes, because:

$$(\boldsymbol{v}\times\boldsymbol{B}^1)\cdot\frac{\partial f_a^0(p)}{\partial \boldsymbol{p}} = 0 .$$

In the case of the Coulomb plasma we get instead of (30.6) - (30.9):

$$\nabla\times\boldsymbol{E}^1 = 0 , \qquad \nabla\cdot\boldsymbol{E}^1 = 4\pi\sum_a e_a n_a \int d\boldsymbol{p}\, f_a^1 \tag{30.10}$$

and from (29.15):

$$\nabla\times\boldsymbol{E}^1 = 0 , \qquad \frac{\partial \boldsymbol{E}^1}{} + 4\pi\sum e_a n_a \int d\boldsymbol{p}\, \boldsymbol{v}\, f_a^1 = 0 \tag{30.11}$$

From here on, we shall omit the superscript 1 on f_a^1, $\boldsymbol{E}^1$, $\boldsymbol{B}^1$.

Sometimes, instead of the differential equations (30.5), we may use the corresponding integral equation. In order to derive the latter, we transfer the last term of (30.5) to the right-hand side, and consider it as a source term. The solution to the equation for f_a can then be represented as the sum of two terms: the solution of the homogeneous equation, corresponding to the initial condition $f_a = f_a(\boldsymbol{r},\boldsymbol{p},0)$ for $t=0$, and the particular solution of the inhomogeneous equation, which vanishes for $t=0$:

$$f_a(\boldsymbol{r},\boldsymbol{p},t) = f_a(\boldsymbol{r}-\boldsymbol{v}t,\ \boldsymbol{p},0) - e_a\int_0^t dt'\,\boldsymbol{E}\,[\boldsymbol{r}-\boldsymbol{v}(t-t'),\ t']\cdot\frac{\partial f_a^0(p)}{\partial\boldsymbol{p}}\ . \tag{30.12}$$

It is easily shown that this equation is equivalent to (30.5). Its right-hand side depends on f_a through $\boldsymbol{E}$, hence it is an integral equation.

From the structure of (30.12) follows that the charge and current densities can be represented in the form:

$$q = q^{\mathbf{ext}} + q^{\mathbf{ind}}\ ,\qquad \boldsymbol{j} = \boldsymbol{j}^{\mathbf{ext}} + \boldsymbol{j}^{\mathbf{ind}} \tag{30.13}$$

The first terms are defined as follows:

$$\begin{aligned} q^{\mathbf{ext}} &= \sum_a e_a n_a \int d\boldsymbol{p}\, f_a(\boldsymbol{r}-\boldsymbol{v}t,\ \boldsymbol{p},0) \\ \boldsymbol{j}^{\mathbf{ext}} &= \sum_a e_a n_a \int d\boldsymbol{p}\,\boldsymbol{v}\, f_a(\boldsymbol{r}-\boldsymbol{v}t,\ \boldsymbol{p},0) \end{aligned} \tag{30.14}$$

they are given quantities (or 'external' quantities), because the integrals only involve the given initial value of $f_a(\boldsymbol{r},\boldsymbol{p},0)$.

The second terms in (30.13) are proportional to the field $\boldsymbol{E}$, hence they are called 'induced' quantities:

$$q^{\mathbf{ind}} = -\sum_a e_a^2 n_a \int d\boldsymbol{p} \int_0^t dt'\,\boldsymbol{E}[\boldsymbol{r}-\boldsymbol{v}(t-t'),\ t']\cdot\frac{\partial f_a^0}{\partial\boldsymbol{p}} \tag{30.15}$$

$$\boldsymbol{j}^{\mathbf{ind}} = -\sum_a e_a^2 n_a \int d\boldsymbol{p}\ \boldsymbol{v} \int_0^t dt'\,\boldsymbol{E}[\boldsymbol{r}-\boldsymbol{v}(t-t'),\ t']\cdot\frac{\partial f_a^0}{\partial\boldsymbol{p}} \tag{30.16}$$

When the expressions (30.14) – (30.16) are substituted into (30.6) – (30.9) we obtain a set of linear inhomogeneous equations for the fields. Their source terms are defined in terms of the external charge and current densities (30.14).

The corresponding equations for $\boldsymbol{E}$ in the case of the Coulomb plasma are obtained by substituting (30.13) into (30.10), (30.11).

We shall study the corresponding homogeneous equations for $\boldsymbol{E}$, $\boldsymbol{B}$, in order to derive the electromagnetic properties of the collisionless plasma.

31. ELECTRICAL CONDUCTIVITY AND DIELECTRIC CONSTANT OF A COLLISIONLESS PLASMA

The expression of the electrical conductivity of a collisionless plasma will be derived as a limiting value of a more general, collisional, expression, in which the collision frequency $\nu_a = \tau_a^{-1} \to 0$ is made to approach zero. To this purpose, we introduce into Eq. (30.5) an additional dissipative term $-\nu_a f_a$ on the right-hand side:

$$\frac{\partial f_a}{\partial t} + \boldsymbol{v}\cdot\frac{\partial f_a}{\partial \boldsymbol{r}} + e_a \boldsymbol{E}\cdot\frac{\partial f_a^0}{\partial \boldsymbol{p}} = -\nu_a f_a \equiv -\nu f_a \tag{31.1}$$

We assumed here that ν_a is independent of the subscript a. In the final expression, we let $\nu \to 0$.

We note that Eq. (31.1) does not make sense for finite ν, as it fails to satisfy the continuity equation. In order to avoid this difficulty we should introduce a dissipative term which vanishes as $\nu \to 0$ but also satisfies the continuity equation for finite ν. However, we need not dwell on this general problem here, as we are only interested in the limit $\nu \to 0$.

As for Eq. (30.5), the solution of (31.1) can be represented as a sum of two terms, similar to (30.12). Here, we shall only consider the terms proportional to the field, i.e. the induced parts of q and $\boldsymbol{j}$.

We look for a solution of (31.1) and (30.6) - (30.9) in the form:

$$f_a(\boldsymbol{r},\boldsymbol{p},t) = f_a(\omega,\boldsymbol{k},\boldsymbol{p}) \exp -i(\omega t - \boldsymbol{kr}) + \text{c.c.}$$

$$\boldsymbol{E}(\boldsymbol{r},t) = \boldsymbol{E}(\omega,\boldsymbol{k}) \exp -i(\omega t - \boldsymbol{kr}) + \text{c.c.}$$

$$\boldsymbol{B}(\boldsymbol{r},t) = \boldsymbol{B}(\omega,\boldsymbol{k}) \exp -i(\omega t - \boldsymbol{kr}) + \text{c.c.} \tag{31.2}$$

Substituting these expressions into (31.1), (30.6) - (30.9) we find a set of equations for the complex amplitudes $f_a(\omega,\boldsymbol{k},\boldsymbol{p})$, $\boldsymbol{E}(\omega,\boldsymbol{k})$, $\boldsymbol{B}(\omega,\boldsymbol{k})$:

$$-i(\omega - \boldsymbol{k}\cdot\boldsymbol{v} + i\nu) f_a = -e_a \boldsymbol{E}\cdot\frac{\partial f_a^0}{\partial \boldsymbol{p}} \tag{31.3}$$

$$i(\boldsymbol{k}\times\boldsymbol{B}) = -\frac{i\omega}{c}\boldsymbol{E} + \frac{4\pi}{c}\boldsymbol{j} \tag{31.4}$$

$$i(\boldsymbol{k}\times\boldsymbol{E}) = \frac{i\omega}{c}\boldsymbol{B} \tag{31.5}$$

$$i(\boldsymbol{k}\cdot\boldsymbol{B}) = 0 \tag{31.6}$$

$$i(\boldsymbol{k}\cdot\boldsymbol{B}) = 4\pi q \tag{31.7}$$

We introduced here the following symbols for the Fourier components of the charge and current densities, $q(\omega,\boldsymbol{k})$, $\boldsymbol{j}(\omega,\boldsymbol{k})$:

$$q = \sum_a e_a n_a \int d\boldsymbol{p}\, f_a(\omega,\boldsymbol{k},\boldsymbol{p}) \tag{31.8}$$

$$\boldsymbol{j} = \sum_a e_a n_a \int d\boldsymbol{p}\, \boldsymbol{v}\, f_a(\omega,\boldsymbol{k},\boldsymbol{p})\,. \tag{31.9}$$

Equations (31.3) - (31.7) being linear, the vectors $\boldsymbol{E}(\omega,\boldsymbol{k})$ and $\boldsymbol{B}(\omega,\boldsymbol{k})$ can be represented as follows:

$$E = \frac{k}{k^2}(k \cdot E) + E - \frac{k}{k^2}(k \cdot E) \equiv E^{\parallel} + E^{\perp}, \quad B = B^{\perp},$$
$$(k \times E^{\parallel}) = 0, \quad k \cdot E^{\perp} = 0. \tag{31.10}$$

The superscripts $\parallel$, $\perp$, denote, respectively, components of the vectors parallel and perpendicular to k. The field $E^{\parallel}$ will be called longitudinal, and the fields $E^{\perp}, B^{\perp}$, transverse. A similar decomposition is performed on the vector j:

$$j = j^{\parallel} + j^{\perp}, \quad k \times j^{\parallel} = 0, \quad k \cdot j^{\perp} = 0. \tag{31.11}$$

Corresponding to this decomposition, the set of equations (31.3) - (31.7) can be split into two subsets. The first involves the longitudinal field $E^{\parallel}$ and the corresponding part of the distribution function, f_{a1} (through Eq. (31.3)). The second involves the transverse fields $E^{\parallel}$, $B^{\perp}$ and the corresponding part of the distribution function, f_{a2}.

The first set of equations is

$$-i(\omega - k \cdot v + i\nu) f_{a1} = -E^{\parallel} \cdot \frac{\partial f_a^0}{\partial p} \tag{31.12}$$

$$ik \cdot E^{\parallel} = 0, \quad ik \cdot E^{\parallel} = 4\pi \sum_a e_a n_a \int dp\, f_{a1}. \tag{31.13}$$

From (31.13) we get

$$E^{\parallel} = -i \frac{k}{k^2} \sum_a 4\pi e_a n_a \int dp\ f_{a1}. \tag{31.14}$$

The second equation (31.13) follows from the Poisson equation. From the Maxwell equation (31.4) we may obtain an alternative form for $E^{\parallel}$. Taking the scalar product of (31.4) with k, we get

$$E^{\parallel} = -\frac{4\pi i}{\omega} j^{\parallel} = -\frac{4\pi i}{\omega} \sum_a e_a n_a \int dp \frac{k}{k^2}(k \cdot v) f_{a1}. \tag{31.15}$$

In the last term, we used the fact that $j^{\parallel} = k(k \cdot j)/k^2$. The set of equations for f_{a2}, E, B is

$$-i(\omega - k \cdot v + i\nu) f_{a2} = -e_a E^{\perp} \cdot \frac{\partial f_a^0}{\partial p} \tag{31.16}$$

$$i(k \times B) = -\frac{i\omega}{c} E^{\perp} + \frac{4\pi}{c} j^{\perp} \tag{31.17}$$

$$i(k \cdot E^{\perp}) = \frac{i\omega}{c} B \tag{31.18}$$

$$ik \cdot B = 0, \quad ik \cdot E = 0. \tag{31.19}$$

Eliminating the vector B from (31.17) we find an equation for $E^{\perp}$:

$$(\omega^2 - c^2 k^2) E^{\perp} = -4\pi i\omega j^{\perp} = -4\pi i\omega \sum_a e_a n_a \int dp \left[v - \frac{k}{k}(k \cdot v)\right] f_{a2} \tag{31.20}$$

where we used the property $\boldsymbol{j}^{\perp} = \boldsymbol{j} - \boldsymbol{k}(\boldsymbol{k}\cdot\boldsymbol{j})/k^2$.

Eliminating now f_{a1} from (31.14) we find an equation for $\boldsymbol{E}^{\|}$, which we write in the form:

$$\varepsilon^{\|}(\omega, \boldsymbol{k})\, \boldsymbol{E}^{\|}(\omega, \boldsymbol{k}) = 0\,, \tag{31.21}$$

where

$$\varepsilon^{\|}(\omega, \boldsymbol{k}) = 1 + \sum_a \frac{4\pi}{k^2} e_a^2 n_a \int d\boldsymbol{p}\, \frac{\boldsymbol{k}\cdot(\partial f_a^0/\partial \boldsymbol{p})}{\omega - \boldsymbol{k}\cdot\boldsymbol{v} + i\nu}\,. \tag{31.22}$$

This function is called the longitudinal dielectric constant of the plasma. Indeed, the Poisson equation for the Fourier-transformed electric induction in the absence of external charges is:

$$i\boldsymbol{k}\cdot\boldsymbol{D}(\omega,\boldsymbol{k}) = i\varepsilon(\omega,\boldsymbol{k})\Big(\boldsymbol{k}\cdot\boldsymbol{E}(\omega,\boldsymbol{k})\Big) = i\varepsilon(\omega,\boldsymbol{k})(\boldsymbol{k}\cdot\boldsymbol{E}^{\|}) = 0 \qquad (31.21)_2$$

The comparison of this equation with (31.21) justifies the name given to $\varepsilon^{\|}$.

Had we used Eq. (31.15) instead of (31.14) for the field $\boldsymbol{E}$, we would have obtained the following expression for $\varepsilon^{\|}(\omega, \boldsymbol{k})$:

$$\varepsilon^{\|}(\omega, \boldsymbol{k}) = 1 + \sum_a \frac{4\pi e_a^2 n_a}{\omega k^2} \int d\boldsymbol{p}\, \frac{(\boldsymbol{k}\cdot\boldsymbol{v})\, \boldsymbol{k}\cdot\partial f_a^0/\partial \boldsymbol{p}}{\omega - \boldsymbol{k}\cdot\boldsymbol{v} + i\nu}\,. \tag{31.23}$$

It is easily seen that this expression is equivalent to (31.22) in the limit $\nu \to 0$.

We now substitute f_{a2} from (31.16) into the last term of (31.20), which becomes:

$$-\omega \sum_a 4\pi e_a^2 n_a \int d\boldsymbol{p}\, \frac{[\boldsymbol{v} - \boldsymbol{k}(\boldsymbol{k}\cdot\boldsymbol{v})/k^2]\,\boldsymbol{E}^{\perp}\cdot(\partial f_a^2/\partial \boldsymbol{p})}{\omega - \boldsymbol{k}\cdot\boldsymbol{v} + i\nu}\,. \tag{31.24}$$

We integrate this expression by parts, noting that the derivative of the denominator yields a vanishing contribution. In calculating the derivative of the numerator, we recall that $\boldsymbol{k}\cdot\boldsymbol{E}^{\perp} = 0$. As a result, (31.24) reduces to

$$\omega \sum_a \frac{4\pi e_a^2 n_a}{m_a} \int d\boldsymbol{p}\, \frac{f_a^0}{\omega - \boldsymbol{k}\cdot\boldsymbol{v} + i\nu}\, \boldsymbol{E}^{\perp}\,. \tag{31.25}$$

We substitute this expression into (31.20) and rewrite the result as:

$$[\omega^2 \varepsilon^{\perp}(\omega, \boldsymbol{k}) - c^2 k^2]\, \boldsymbol{E}^{\perp}(\omega, \boldsymbol{k}) = 0 \tag{31.26}$$

where:

$$\varepsilon^{\perp}(\omega, \boldsymbol{k}) = 1 - \sum_a \frac{4\pi e_a^2 n_a}{\omega m_a} \int d\boldsymbol{p}\, \frac{f_a^0(p)}{\omega - \boldsymbol{k}\cdot\boldsymbol{v} + i\nu}\,. \tag{31.27}$$

From (31.26) it is clear that $\varepsilon^{\perp} = n^2$, where n is the refractive index for waves of frequency ω and wave vector $\boldsymbol{k}$. Thus $\varepsilon^{\perp}$ is the transverse dielectric constant of the plasma.

If we do not perform the decomposition of the set (31.3)-(31.7) into longitudinal and transverse parts, we can still find a general relation between the vectors $\boldsymbol{D}$ and $\boldsymbol{E}$. This relation is of a tensorial nature; thus the electromagnetic properties of the plasma will be characterized by dielectric tensor $\varepsilon_{ij}(\omega,\boldsymbol{k})$. We shall derive an expression for this tensor and show that it is wholly defined in terms of the scalars $\varepsilon^{\parallel}$ and $\varepsilon^{\perp}$.

We substitute f_a from (31.3) into (31.9) and write the result in the form:

$$j_i(\omega,\boldsymbol{k}) = \sigma_{ij}(\omega,\boldsymbol{k})\, E_j(\omega,\boldsymbol{k}). \tag{31.28}$$

We introduced here the electrical conductivity tensor of the collisionless plasma:

$$\sigma_{ij}(\omega,\boldsymbol{k}) = -i \sum_a e_a^2 n_a \int d\boldsymbol{p}\, \frac{v_i\,(\partial f_a^0/\partial p_j)}{\omega - \boldsymbol{k}\cdot\boldsymbol{v} + i\nu}. \tag{31.29}$$

In order to define the dielectric tensor, we note that, in the absence of external currents, the right-hand side of (31.4) can be represented in terms of the electric induction vector $\boldsymbol{D}(\omega,\boldsymbol{k})$:

$$-\frac{i\omega}{c}\boldsymbol{D}(\omega,\boldsymbol{k}) \equiv -\frac{i\omega}{c}\boldsymbol{E}(\omega,\boldsymbol{k}) + \frac{4\pi}{c}\,\boldsymbol{j}(\omega,\boldsymbol{k}). \tag{31.30}$$

On the other hand, by definition of the tensor ε_{ij}:

$$D_i(\omega,\boldsymbol{k}) = \varepsilon_{ij}(\omega,\boldsymbol{k})\, E_j(\omega,\boldsymbol{k}) \tag{31.31}$$

From Eqs (31.28), (31.30), (31.31) we find the relation

$$\varepsilon_{ij} = \delta_{ij} + \frac{4\pi i}{\omega}\,\sigma_{ij}. \tag{31.32}$$

Using (31.29) we obtain the explicit form of the dielectric tensor of the collisionless plasma:

$$\varepsilon_{ij}(\omega,\boldsymbol{k}) = \delta_{ij} + \sum_a \frac{4\pi e_a^2 n_a}{\omega} \int d\boldsymbol{p}\, \frac{v_i(\partial f_a^0/\partial p_j)}{\omega = \boldsymbol{k}\cdot\boldsymbol{v} + i\nu}. \tag{31.33}$$

In the absence of an external field, the right-hand side depends on a single vector, $\boldsymbol{k}$. The tensor ε_{ij} is therefore defined by two functions of ω and $|\boldsymbol{k}|$, which can be chosen to be $\varepsilon^{\parallel}$, $\varepsilon^{\perp}$. Indeed, ε_{ij} can always be represented as

$$\varepsilon_{ij} = \frac{k_i k_j}{k^2}\,\varepsilon^{\parallel} + \left(\delta_{ij} - \frac{k_i k_j}{k^2}\right)\varepsilon^{\perp} \tag{31.34}$$

with

$$\varepsilon^{\parallel} = \frac{k_i k_j}{k^2}\,\varepsilon_{ij}, \qquad \varepsilon^{\perp} = \frac{1}{2}\left(\delta_{ij} - \frac{k_i k_j}{k^2}\right)\varepsilon_{ij}. \tag{31.35}$$

Substituting (31.33) into the first of these equations we obtain an expression coinciding with (31.23). The second equation (31.35) similarly yields:

$$\varepsilon^{\perp}(\omega, \boldsymbol{k}) = 1 + \sum_a \frac{2\pi e_a^2 n_a}{\omega k^2} \int d\boldsymbol{p} \, \frac{[(\boldsymbol{k}\times\boldsymbol{v})\times\boldsymbol{k}]\cdot(\partial f_a^0/\partial \boldsymbol{p})}{\omega - \boldsymbol{k}\cdot\boldsymbol{v} + i\nu} \tag{31.36}$$

where we used the identity:

$$(\boldsymbol{k}\times\boldsymbol{v})\times\boldsymbol{k} = k^2\boldsymbol{v} - (\boldsymbol{k}\cdot\boldsymbol{v})\boldsymbol{k} \tag{31.37}$$

An integration by parts proves the equivalence of (31.36) with (31.27).

The functions $\varepsilon^{\parallel}, \varepsilon^{\perp}$ depend only on the absolute value k, not on the complete vector $\boldsymbol{k}$; this is due to the fact that the function f_a^0 depends only on the modulus p.

We now use the expressions of the dielectric tensor and of the electrical conductivity for an analysis of the wave properties of the collisionless plasma.

32. WAVE PROPERTIES OF A COLLISIONLESS PLASMA

In the absence of external charges and currents, Eqs (31.21) and (31.26) are homogeneous. They possess non-trivial solutions whenever ω and $\boldsymbol{k}$ are related by the dispersion equations:

$$\varepsilon^{\parallel}(\omega, \boldsymbol{k}) = 0 \tag{32.1}$$

$$\omega^2\varepsilon^{\perp}(\omega, \boldsymbol{k}) - c^2k^2 = 0 , \tag{32.2}$$

(32.1) is the dispersion relation for longitudinal waves, and (32.2) the dispersion relation for transverse waves.

If the electromagnetic properties of the plasma are described by the dielectric tensor ε_{ij}, (32.1), (32.2) can be grouped in a single dispersion equation:

$$\left| \frac{\omega^2}{c^2}\varepsilon_{ij}(\omega, \boldsymbol{k}) - \delta_{ij}k^2 + k_i k_j \right| = 0 . \tag{32.3}$$

This form is more general, because it holds even when the plasma is anisotropic (e.g., in the presence of external fields), and the tensor ε_{ij} is not of the simple form (31.34).

The dispersion equation determines the relation between the frequencies and the wavevectors of the longitudinal and transverse eigenmodes of the plasma. If there are waves with real $\boldsymbol{k}$ in the plasma (for instance in an infinite plasma, or in a plasma contained in a resonator), the dispersion equation yields the dependence $\omega = \omega(\boldsymbol{k})$. In general, the eigenfrequencies will be complex:

$$\omega = \omega' + i\omega'' \equiv \omega' - i\gamma . \tag{32.4}$$

We thus obtain from the dispersion equation two functional relations:

$$\omega' = \omega'(\boldsymbol{k}) \tag{32.5}$$

$$\gamma = \gamma(\boldsymbol{k}) \tag{32.6}$$

defining the frequencies ω' and the damping rates γ of the eigenmodes, as functions of $\boldsymbol{k}$.

If the plasma is inhomogeneous, e.g., because of the action of the boundaries, there may exist stationary states for which ω is real. From the dispersion equation we find in that case the dependence on ω of the projection of $\boldsymbol{k}$ on an arbitrary axis.

Let us consider the case of weak damping:

$$\gamma \ll \omega' . \tag{32.7}$$

In that case we find from (32.1):

$$\varepsilon^{\parallel}(\omega' - i\gamma, \boldsymbol{k}) \equiv \varepsilon^{\parallel}(\omega', \boldsymbol{k}) - i\gamma \frac{\partial \varepsilon^{\parallel}}{\partial \omega'} = 0 . \tag{32.8}$$

We see that the dispersion relation involves the function $\varepsilon^{\parallel}(\omega', k)$ for real values of ω' and $\boldsymbol{k}$. It is a complex-valued function, defined by (31.22):

$$\varepsilon^{\parallel}(\omega', \boldsymbol{k}) = \mathrm{Re}\, \varepsilon^{\parallel}(\omega', \boldsymbol{k}) + \mathrm{Im}\, \varepsilon^{\parallel}(\omega', \boldsymbol{k}) . \tag{32.9}$$

We thus obtain

$$\mathrm{Re}\, \varepsilon^{\parallel}(\omega', \boldsymbol{k}) = 1 + \sum_a \frac{4\pi e_a^2 n_a}{k^2} P \int d\boldsymbol{p} \, \frac{\boldsymbol{k} \cdot (\partial f_a^0 / \partial \boldsymbol{p})}{\omega' - \boldsymbol{k} \cdot \boldsymbol{v}} \tag{32.10}$$

$$\mathrm{Im}\, \varepsilon^{\parallel}(\omega', \boldsymbol{k}) = -\sum_a \frac{4\pi e_a^2 n_a}{k^2} \int d\boldsymbol{p} \; \delta(\omega' - \boldsymbol{k} \cdot \boldsymbol{v}) \, \boldsymbol{k} \cdot \frac{\partial f_a^0}{\partial \boldsymbol{p}} . \tag{32.11}$$

We used here the relation:

$$\lim_{\nu \to 0} \frac{1}{x + i\nu} = P \frac{1}{x} - i\pi\delta(x) . \tag{32.12}$$

The symbol P means that the integration over the singularity at the point $x = 0$ must be understood as a principal value.

Because of the condition (32.7), we may retain in the second term of (32.8) only the contribution of $\mathrm{Re}\, \varepsilon^{\parallel}$. We thus get, to first order in γ/ω', the two equations:

$$\mathrm{Re}\, \varepsilon^{\parallel}(\omega', \boldsymbol{k}) = 0 , \tag{32.13}$$

$$\gamma = \frac{\mathrm{Im}\, \varepsilon^{\parallel}(\omega', \boldsymbol{k})}{\partial\, \mathrm{Re}\, \varepsilon^{\parallel}(\omega', \boldsymbol{k}) / \partial \omega'} . \tag{32.14}$$

The first equation provides us with the relation $\omega' = \omega'(\boldsymbol{k})$. Knowing the latter, we find the damping rate $\gamma(\boldsymbol{k})$ from the second equation.

Performing similar transformations on the dispersion equation for transverse waves, we find the following two equations from (32.2):

$$\omega'^2 \ \mathrm{Re}\, \varepsilon^{\perp}(\omega', \boldsymbol{k}) - c^2 k^2 = 0 , \tag{32.15}$$

$$\gamma = \frac{\omega'^2 \ \mathrm{Im}\, \varepsilon^{\perp}(\omega', \boldsymbol{k})}{\partial[\omega'^2 \ \mathrm{Re}\, \varepsilon^{\perp}]/\partial\omega'} \tag{32.16}$$

where

$$\mathrm{Re}\, \varepsilon^{\perp} = 1 - \sum_a \frac{4\pi e_a^2 n_a}{\omega' m_a} P \int d\boldsymbol{p} \, \frac{f_a^0}{\omega' - \boldsymbol{k}\cdot\boldsymbol{v}} \tag{32.17}$$

$$\mathrm{Im}\, \varepsilon^{\perp} = \sum_a \frac{4\pi e_a^2 n_a}{\omega' m_a} \int d\boldsymbol{p} \, \delta(\omega' - \boldsymbol{k}\cdot\boldsymbol{v}) \, f_a^0 . \tag{32.18}$$

We now consider several concrete examples.

1. The cold plasma ($T_a = 0$). In this case:

$$f_a^0 = \delta(\boldsymbol{p}) . \tag{32.19}$$

From (32.13), (32.14) we find

$$\omega'^2 = \omega_L^2 , \qquad \gamma = 0 , \tag{32.20}$$

where ω_L is the Langmuir frequency of the plasma:

$$\omega_L = \left(\sum_a \frac{4\pi e_a^2 n_a}{m_a} \right)^{\frac{1}{2}} . \tag{32.21}$$

We see that in a cold plasma there may exist undamped oscillations of frequency ω_L. This frequency is independent of the wavevector.

2. Waves in an electron plasma.

$$f_e^0 = (2\pi m_e k_B T_e)^{-\frac{3}{2}} \exp\left[-p^2/2 m_e k_B T_e\right] , \qquad f_i^0 = \delta(\boldsymbol{p}) . \tag{32.22}$$

If $\omega/k \gg v_{Te}$, (i.e., the phase velocity of the waves is much larger than the thermal speed of the electrons), we find from (32.13), (32.14), (32.10) and (32.11)

$$\omega'^2 = \omega_e^2 (1 + 3 r_e^2 k^2) \tag{32.23}$$

$$\gamma = \left(\frac{\pi}{8}\right)^{\frac{1}{2}} \frac{\omega_e}{r_e^3 k^3} \exp(-1/2 r_e^2 k^2) \tag{32.24}$$

where

$$\omega_e = (4\pi e^2 n_e / m_e)^{\frac{1}{2}} ,$$

$$r_e = v_{Te}/\omega_e = (k_B T_e / 4\pi e^2 n_e)^{\frac{1}{2}}$$

are, respectively, the Langmuir frequency and the Debye radius for the electrons. The damping rate γ defined by (32.24) is called the Landau damping rate. From

(32.11) we see that the damping rate is related to

$$\partial f_e^0(p_x)/\partial p_x\Big|_{v_x = \omega'/k}, \qquad f_e(p_x) = \int dp_y\, dp_z\, f_e(\boldsymbol{p}) \tag{32.25}$$

i.e., to the derivative of the function $f_e(p_x)$ at a value of v_x equal to the phase velocity of the wave.

3. Longitudinal waves in a non-isothermal plasma, when $T_e \gg T_i$, for the range of phase velocities: $v_{Te} \gg \omega'/k \gg v_{Ti}$. In this case

$$\omega'^2 = \frac{\omega_i^2 r_e^2 k^2}{1 + r_e^2 k^2}$$

$$\gamma = \left(\frac{\pi}{8}\right)^{\frac{1}{2}} \frac{\omega_i^2 r_e^2 k^2}{(k_B T_e/m_e)^{\frac{1}{2}} k (1 + r_e^2 k^2)^2} \tag{32.26}$$

where ω_i is the ion Langmuir frequency. For long wavelengths $(kr_e \ll 1)$ these expressions reduce to:

$$\omega'^2 = k^2 v_s^2, \qquad \gamma = \left(\frac{\pi}{8}\frac{m_e}{m_i}\right)^{\frac{1}{2}} k v_s \tag{32.27}$$

where

$$v_s = \left(\frac{k_B T_e}{m_i}\right)^{\frac{1}{2}} \tag{32.28}$$

is the sound speed in the collisionless non-isothermal plasma. From (32.27) we find the ratio γ/ω' for the ion-acoustic waves:

$$\frac{\gamma}{\omega'} = \left(\frac{\pi}{8}\frac{m_e}{m_i}\right)^{\frac{1}{2}} \ll 1 .$$

In the opposite limit of short wavelengths, $kr_e \gg 1$, we find from (32.26)

$$\omega'^2 = \omega_i^2, \qquad \gamma = \left(\frac{\pi}{8}\right)^{\frac{1}{2}} \left(\frac{m_e}{k_B T_e}\right)^{\frac{1}{2}} \frac{\omega_i^2}{r_e^2 k^3} \tag{32.29}$$

These are oscillations whose frequencies equal the ion Langmuir frequency.

4. Transverse waves in a Maxwellian plasma, for $\omega/k \gg v_T$. If $v_T/c \ll 1$, i.e., in a non-relativistic plasma, Eqs (32.15) yield:

$$\omega'^2 = \omega_L^2 + c^2 k^2, \quad \gamma = 0. \tag{32.30}$$

The vanishing of the damping rate of transverse waves can be understood as follows. From (32.18) follows that $\mathrm{Im}\,\varepsilon^{\perp}$ is proportional to $f_a(p_x)$, (where $\boldsymbol{x} \parallel \boldsymbol{k}$), for $p_x = m\omega'/k$. However, from (32.30) we see that the phase velocity is

$$(\omega'/k) = c(1 + \omega^2/c^2 k^2)^{\frac{1}{2}} > c \tag{32.31}$$

thus

$$f^0(p_x)\Big|_{v_x = \omega'/k} = 0 ,$$

hence the damping rate vanishes.

5. The screened static field in a plasma. For $\omega = 0$ and a Maxwellian distribution, we find from (32.1) and (32.10):

$$1 + \frac{1}{r_D^2 k^2} = 0 , \qquad k^2 = -\frac{1}{r_D^2} . \tag{32.32}$$

In the one-dimensional case, this dispersion equation corresponds to the following equation for the electric potential $\phi(x)$:

$$\frac{d^2\phi}{dx^2} - \frac{\phi}{r_D^2} = 0 . \tag{32.33}$$

The solution of this equation with the boundary conditions: $\phi = \phi_0$ for $x = 0$, and $\phi = 0$ for $x = \infty$, is

$$\phi(x) = \phi_0 \exp(-x/r_D) . \tag{32.34}$$

It follows that the static field in the collisionless plasma is screened at a distance of the order of the Debye length.

We discussed here a few examples which will be useful below. The detailed investigation of the wave processes in a plasma is outside the scope of this book. The study of these processes can be found in many books [8 - 28].

Further below, a few additional results of the theory of waves in plasmas will be needed. They will be introduced at the appropriate places.

CHAPTER 7

Kinetic Equations for the Ideal Fully Ionized Plasma

33. LIMITATIONS NECESSARY FOR THE DERIVATION OF KINETIC EQUATIONS

According to (26.3), the collision integral I_a for a Coulomb plasma is

$$I_a(x,t) = -\frac{e_a}{n_a} \cdot \left\langle \delta N_a \; \delta \boldsymbol{E} \right\rangle_{x,\boldsymbol{r},t} \quad . \tag{33.1}$$

We use the spectral representation for the second moment of two arbitrary random functions $A(\boldsymbol{r},t)$, $B(\boldsymbol{r}',t)$:

$$\begin{aligned} \left\langle AB \right\rangle_{\boldsymbol{r},t,\boldsymbol{r}',t'} &\equiv \left\langle AB \right\rangle_{\boldsymbol{r}-\boldsymbol{r}',\; t-t',\; \boldsymbol{r},t} \\ &= (2\pi)^{-4} \int d\omega \, d\boldsymbol{k} \, (AB)_{\omega,\boldsymbol{k},t,\boldsymbol{r}} \exp\left[-i\omega(t-t') + i\boldsymbol{k}\cdot(\boldsymbol{r}-\boldsymbol{r}')\right] \end{aligned} \tag{33.2}$$

As $\left\langle AB \right\rangle_{\boldsymbol{r},t,\boldsymbol{r}',t'}$ is a real function, $(AB)_{\omega,\boldsymbol{k},t,\boldsymbol{r}}$ has the property

$$(AB)_{\omega,\boldsymbol{k},t,\boldsymbol{r}} = (AB)^{*}_{-\omega,-\boldsymbol{k},t,\boldsymbol{r}} \quad . \tag{33.3}$$

From (33.2), for $t=t'$, $\boldsymbol{r}=\boldsymbol{r}'$, we find †

$$\left\langle AB \right\rangle_{\boldsymbol{r},\boldsymbol{r},t,t} = (2\pi)^{-4} \int d\omega \, d\boldsymbol{k} \; \mathrm{Re}\,(AB)_{\omega,\boldsymbol{k},t,\boldsymbol{r}} \quad . \tag{33.4}$$

Using this equation, we rewrite the collision integral as follows

$$\begin{aligned} I_a(x,t) &= -\frac{e_a}{(2\pi)^4 n_a} \frac{\partial}{\partial \boldsymbol{p}} \cdot \int d\omega \; d\boldsymbol{k} \; \mathrm{Re}\,(\delta N_a \; \delta \boldsymbol{E})_{\omega,\boldsymbol{k},\boldsymbol{r},\boldsymbol{p},t} \\ &\equiv -\frac{e_a}{n_a (2\pi)^3} \frac{\partial}{\partial \boldsymbol{p}} \cdot \int d\boldsymbol{k} \; \mathrm{Re}\,(\delta N_a \; \delta \boldsymbol{E})_{\boldsymbol{k},\boldsymbol{r},\boldsymbol{p},t} \end{aligned} \tag{33.5}$$

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† **Note that** $\mathrm{Im}\,(AB)_{\omega,\boldsymbol{k},t,\boldsymbol{r}}$ **is odd in the variables** $\omega,\boldsymbol{k}$.

In order to derive a kinetic equation for f_a we thus must express the spectral function $(\delta N_a\ \delta \boldsymbol{E})_{\omega,\boldsymbol{k},\boldsymbol{r},\boldsymbol{p},t}$ in terms of f_a. To find such an expression in the polarization approximation, (see sect. 27), we must either solve the set of equations for the second moments $\langle \delta N_a\ \delta N_b \rangle_{x,x',t}$, $\langle \delta N_a\ \delta N_b \rangle_{x,x',t,t'}$ or solve the set of equations (27.7), (27.8) for the random functions δN_a. The latter method, which is much simpler, will be used here

We rewrite (27.7), (27.8) explicitly for a Coulomb plasma

$$\left(\frac{\partial}{\partial t} + \boldsymbol{v}\cdot\frac{\partial}{\partial \boldsymbol{r}} + \boldsymbol{F}_a\cdot\frac{\partial}{\partial \boldsymbol{p}}\right)\left(\delta N_a - \delta N_a^{\text{source}}\right) = -\, e_a\, \delta \boldsymbol{E}\cdot\frac{\partial n_a f_a}{\partial \boldsymbol{p}} ,$$

$$\nabla \times \delta \boldsymbol{E} = 0 , \qquad \nabla \cdot \delta \boldsymbol{E} = 4\pi \sum_a e_a \int d\boldsymbol{p}\ \delta N_a . \tag{33.6}$$

The source correlation function is defined by (27.8)

$$\left(\frac{\partial}{\partial t} + \boldsymbol{v}\cdot\frac{\partial}{\partial \boldsymbol{r}} + \boldsymbol{F}_a\cdot\frac{\partial}{\partial \boldsymbol{p}}\right)\langle \delta N_a\ \delta N_b \rangle^{\text{source}}_{x,t,x',t'} = 0 . \tag{33.7}$$

In (33.6), (33.7) the average force $\boldsymbol{F}_a$ is defined by

$$\boldsymbol{F}_a = e_a \boldsymbol{E}_0 + \frac{e_a}{c}(\boldsymbol{v}\times \boldsymbol{B}_0) - e_a \frac{\partial}{\partial \boldsymbol{r}}\cdot \sum_b \int dx' \frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|} f_b(x',t) \tag{33.8}$$

Grouping the first and the last term we can also write

$$\boldsymbol{F}_a = e_a \boldsymbol{E} + \frac{e_a}{c}(\boldsymbol{v}\times \boldsymbol{B}_0) \tag{33.9}$$

where $\boldsymbol{E}$ is the total average field, due to external $(\boldsymbol{E}_0)$ and internal sources.

If instead of the Poisson equation one uses the equation for the total current, then the equation for δE is written as

$$\nabla \times \delta \boldsymbol{E} = 0 , \quad \frac{\partial\, \delta \boldsymbol{E}}{\partial t} + 4\pi \sum_a e_a \int d\boldsymbol{p}\, \boldsymbol{v}\, \delta N_a = 0 . \tag{33.10}$$

In order to define the collision integral it will be sufficient to know the spectral functions in the frequency domain $\omega \gg \nu_a$ and in the range of wavevectors $k \gg l^{-1}$, thus the initial conditions for (33.7) can be defined in the form (27.10)

$$\langle \delta N_a\ \delta N_b \rangle^{\text{source}}_{x,t,x',t'}\Big|_{t=t'} = n_a\, \delta_{ab}\ \delta(x-x')\ f_a(x',t) \tag{33.11}$$

In a plasma, as in a gas, one needs for the derivation of kinetic equations some well-defined assumptions about the character of the relaxation of the correlation or spectral functions.

We denote the relaxation times of the distribution function f_a by τ_a, and the least relaxation time by τ_{rel} ; $l_{\text{rel}} = v_T \tau_{\text{rel}}$ is the corresponding length. In section 29 it was already stated that $l_{\text{rel}} \sim r_D/\mu$, $\tau_{\text{rel}} \sim 1/\omega_e \mu$ and thus, for a dilute plasma, when $\mu \ll 1$,

$$l_{\mathbf{rel}} \sim r_D/\mu \gg r_D, \qquad \tau_{\mathbf{rel}} \sim 1/\omega_e \mu \gg 1/\omega_e . \tag{33.12}$$

As the collision integral is expressed in terms of spectral functions, we must compare $l_{\mathbf{rel}}$, $\tau_{\mathbf{rel}}$ with the correlation length and time $l_{\mathbf{cor}}$, $\tau_{\mathbf{cor}}$.

In a state close to equilibrium, $r_{\mathbf{cor}}$ is defined as the length over which the field of a charged particle is screened, i.e., the Debye radius $r_{\mathbf{D}}$ (see sect. 32). This means that the main contribution comes from a range of wavevectors $k > r_{\mathbf{D}}^{-1}$, i.e., $k_{\mathbf{min}} \sim r_{\mathbf{D}}^{-1}$.

The correlation time for different spatial Fourier components is different:

$$\tau_{\mathbf{cor}} \sim \tau_{\mathbf{cor}}(k) . \tag{33.13}$$

One of the assumptions necessary for the derivation of kinetic equations is that the spectral functions for $k < k_{\mathbf{min}}$ give a negligibly small contribution to the collision integral, [see (10.3)].

The correlation time $\tau_{\mathbf{cor}}(k)$ usually decreases with increasing k; the previous assumption then means that, in deriving the collision integral, one only needs to consider fluctuations with correlation times such that

$$\tau_{\mathbf{cor}}(k) \ll \tau_{\mathbf{max}} = \tau_{\mathbf{cor}}(k_{\mathbf{max}}) . \tag{33.14}$$

For systems close to equilibrium:

$$\tau_{\mathbf{max}} \sim \frac{1}{k_{\mathbf{min}} v_T} \sim \frac{r_{\mathbf{D}}}{v_T} \sim \frac{1}{\omega_e} . \tag{33.15}$$

Combining this result with (33.12) we find that, for $\mu \ll 1$, the assumption corresponds to assuming

$$\tau_{\mathbf{max}} = \tau_{\mathbf{cor}}(k_{\mathbf{min}}) \ll \tau_{\mathbf{rel}} , \qquad k_{\mathbf{min}}^{-1} \sim r_{\mathbf{D}} \ll l_{\mathbf{rel}} . \tag{33.16}$$

As a result of these inequalities, in solving the equations for the correlation functions, or the corresponding equations for δN_a, $\delta \boldsymbol{E}$, one may neglect the initial correlations.

Thus, as in the case of kinetic gas theory, in the derivation of the kinetic equations for plasmas, one assumes that the long range fluctuations, with $\tau_{\mathbf{cor}} \gtrsim \tau_{\mathbf{rel}}$ and $l_{\mathbf{cor}} \gtrsim l_{\mathbf{rel}}$, do not significantly contribute to these equations.

As in the theory of gases (sect. 16) we establish the limit separating short-range and long-range fluctuations from the condition of continuity of the change in the functions resulting from collisions. As in section 16, we denote by l_ϕ^3, τ_ϕ a volume and a time interval which can be considered as physically infinitesimal in the application of kinetic theory. Thus, $\tau_\phi \sim l_\phi / v_T$. By analogy with (16.2)

$$\frac{\tau_{\mathbf{rel}}}{n\, l_\phi^3} = \tau_\phi = \frac{l_\phi}{v_T} , \quad \text{hence} \quad l_\phi^4 = \frac{\tau_{\mathbf{rel}}\, v_T}{n} = \frac{l_{\mathbf{rel}}}{n} . \tag{33.17}$$

As $l_{\mathbf{rel}} \sim r_{\mathbf{D}}/\mu$ and $\mu = 1/n\, r_{\mathbf{D}}^3$,

$$l_\phi = (l_{\mathbf{rel}}/n)^{\frac{1}{4}} \sim r_{\mathbf{D}} . \tag{33.18}$$

The number of particles in the physically infinitesimal volume is

$$n\, l_\phi^3 \sim n\, r_{\mathbf{D}}^3 = 1/\mu \gg 1 . \tag{33.19}$$

Thus, we may take $r_{\mathbf{D}}$ as a limit between long-range and short-range fluctuations. The corresponding time interval is

$$\tau_\phi \sim r_{\mathbf{D}}/v_T \sim 1/\omega_e . \tag{33.20}$$

This choice agrees with condition (33.15), (33.16).

Giving up the condition of complete weakening of initial correlations leads, as in the theory of gases, to the possibility of a kinetic theory of fluctuations (see Chapter 11).

In the kinetic theory of plasmas, when the equations for the spectral functions, or those for the fluctuations δN_a, $\delta \boldsymbol{E}$ are solved, the time delay and the spatial variation of f_a can be neglected over intervals $r_{\mathbf{D}}/v_T \sim 1/\omega_e$ and $r_{\mathbf{D}}$ respectively.

The introduction of a boundary between long-range and short-range fluctuations allows one to give a meaning to the words 'strong', 'weak', 'slow' fields in connection with the solution of the equations for δN_a, $\delta \boldsymbol{E}$.

An electric field is weak when

$$\frac{e E\, l_\phi}{k_{\mathbf{B}} T} \sim \frac{e E\, r_{\mathbf{D}}}{k_{\mathbf{B}} T} \ll 1 . \tag{33.21}$$

A magnetic field is weak when

$$\Omega_L = \frac{eB}{mc} \ll \tau_\phi^{-1} \sim \omega_e . \tag{33.22}$$

An electric field entering the collision integral I_a is considered slowly varying when its frequency ω_0 satisfies

$$\omega_0 \ll \tau_\phi^{-1} \sim \omega_e . \tag{33.23}$$

When the average fields $\boldsymbol{E}, \boldsymbol{B}$ are weak in the sense (33.21), (33.22), then the following simpler equations can be used for the calculation of the spectral functions [29]:

$$\left(\frac{\partial}{\partial t} + \boldsymbol{v} \cdot \frac{\partial}{\partial \boldsymbol{r}}\right) \left(\delta N_a - \delta N_a^{\mathbf{source}}\right) = - e_a\, \delta \boldsymbol{E} \cdot \frac{\partial\, n_a f_a}{\partial \boldsymbol{p}} \tag{33.24}$$

$$\left(\frac{\partial}{\partial t} + \boldsymbol{v} \cdot \frac{\partial}{\partial \boldsymbol{r}}\right) \left\langle \delta N_a\ \delta N_b \right\rangle_{x, t, x', t'}^{\mathbf{source}} = 0 \tag{33.25}$$

These are the equations used in the next sections.

34. SPECTRAL DENSITIES OF THE SOURCE FLUCTUATIONS IN THE IDEAL PLASMA

We look for an expression for the space-time spectral density

$$(\delta N_a\, \delta N_b)^{\text{source}}_{\omega,\boldsymbol{k},\boldsymbol{p},\boldsymbol{p}'} .$$

We start from Eq. (33.25). In order to eliminate directly the correlations with $\tau_{\mathbf{cor}} > \tau_\phi \sim \omega_e^{-1}$, we introduce in (33.25) a dissipative term $\Delta\langle\delta N_a\, \delta N_b\rangle^{\text{source}}$, assuming $\tau^{-1}_{\mathbf{rel}} \ll \Delta \ll \tau^{-1}_\phi$. As τ_ϕ is physically infinitesimal in relation to the function f_a, we let $\Delta \to 0$ in the final result. Thus (33.25) becomes

$$\left(\frac{\partial}{\partial t} + \boldsymbol{v}\cdot\frac{\partial}{\partial \boldsymbol{r}} + \Delta\right)\langle\delta N_a\, \delta N_b\rangle^{\text{source}}_{x,t,x',t'} = 0 \qquad (34.1)$$

which must be solved with the initial condition (33.11).

According to the discussion of section 33, for deriving the kinetic equation we need the spectral densities for

$$\tau_{\mathbf{cor}} \leqslant \omega_e^{-1} \ll \tau_{\mathbf{rel}}, \qquad r_{\mathbf{cor}} \leqslant r_{\mathbf{D}} \ll l_{\mathbf{rel}} . \qquad (34.2)$$

In constructing the kinetic equations for ideal plasmas, the time delay and the spatial variation of f_a are neglected. This means that we consider the zeroth approximation in the parameters $\tau_{\mathbf{cor}}/\tau_{\mathbf{rel}}$, $r_{\mathbf{cor}}/l_{\mathbf{rel}}$. In other words, to calculate the spectral densities, we may consider the function f_a as independent of $\boldsymbol{r}$ and t.

The solution of (34.1) with the initial condition (33.11) is

$$\langle\delta N_a\, \delta N_b\rangle^{\text{source}}_{x,t,x',t'} = \delta_{ab}\, \delta(\boldsymbol{r}-\boldsymbol{r}'-\boldsymbol{v}(t-t'))\,\delta(\boldsymbol{p}-\boldsymbol{p}')\, e^{-\Delta(t-t')}\, n_b\, f_b(x',t') . \qquad (34.3)$$

We introduce the change of variables

$$\boldsymbol{r}-\boldsymbol{r}' = \boldsymbol{r}, \qquad t-t' = \tau . \qquad (34.4)$$

For stationary and homogeneous processes, (34.3) reduces to:

$$\langle\delta N_a\, \delta N_b\rangle^{\text{source}}_{\boldsymbol{r},\tau,\boldsymbol{p},\boldsymbol{p}'} = n_a\, \delta_{ab}\, \delta(\boldsymbol{r}-\boldsymbol{v}\tau)\, \delta(\boldsymbol{p}-\boldsymbol{p}')\, f_a(\boldsymbol{p})\, e^{-\Delta\tau} . \qquad (34.5)$$

We introduce the Fourier transform with respect to $\boldsymbol{r}$:

$$\langle\delta N_a\, \delta N_b\rangle_{\boldsymbol{r},\tau,\boldsymbol{p},\boldsymbol{p}'} = (2\pi)^{-3}\int d\boldsymbol{k}\, (\delta N_a\, \delta N_b)_{\boldsymbol{k},\tau,\boldsymbol{p},\boldsymbol{p}'}\, e^{i\boldsymbol{k}\cdot\boldsymbol{r}} . \qquad (34.6)$$

We thus obtain the spatial spectral density from (34.5):

$$(\delta N_a\, \delta N_b)^{\text{source}}_{\boldsymbol{k},\tau,\boldsymbol{p},\boldsymbol{p}'} = n_a\, \delta_{ab}\, \delta(\boldsymbol{p}-\boldsymbol{p}')\, e^{-\Delta\tau - i\boldsymbol{k}\cdot\boldsymbol{v}\tau}\, f_a(\boldsymbol{p}) . \qquad (34.7)$$

In order to calculate the space-time spectral density, we first introduce the function

$$(\delta N_a\, \delta N_b)^{+}_{\omega,\boldsymbol{k},\boldsymbol{p},\boldsymbol{p}'} \int_0^\infty d\tau\, e^{i\omega\tau}\, (\delta N_a\, \delta N_b)_{\boldsymbol{k},\tau,\boldsymbol{p},\boldsymbol{p}'} \qquad (34.8)$$

and then use the relation :

$$(\delta N_a \delta N_b)_{\omega,\boldsymbol{k},\boldsymbol{p},\boldsymbol{p}'} = (\delta N_a \delta N_b)^{+}_{\omega,\boldsymbol{k},\boldsymbol{p},\boldsymbol{p}'} + \left[(\delta N_a \delta N_b)^{+}_{\omega,\boldsymbol{k},\boldsymbol{p},\boldsymbol{p}'}\right]^{*}_{\substack{a \rightleftharpoons b \\ \boldsymbol{p} \rightleftharpoons \boldsymbol{p}'}} \tag{34.9}$$

valid for stationary processes. From (34.7), (34.8) we obtain

$$(\delta N_a \delta N_b)^{+\,\mathrm{source}}_{\omega,\boldsymbol{k},\boldsymbol{p},\boldsymbol{p}'} = n_a \delta_{ab}\, \delta(\boldsymbol{p}-\boldsymbol{p}')\, \frac{i}{\omega - \boldsymbol{k}\cdot\boldsymbol{v} + i\Delta}\, f_a(\boldsymbol{p}). \tag{34.10}$$

From (34.9), (34.10), in the limit $\Delta \to 0$, we obtain the desired expression for the space-time spectral density :

$$(\delta N_a \delta N_b)^{\mathrm{source}}_{\omega,\boldsymbol{k},\boldsymbol{p},\boldsymbol{p}'} = n_a \delta_{ab}\, \delta(\boldsymbol{p}-\boldsymbol{p}')\, 2\pi\, \delta(\omega - \boldsymbol{k}\cdot\boldsymbol{v})\, f_a(\boldsymbol{p}) . \tag{34.11}$$

We now look for the spectral densities $(\delta N_a \delta \boldsymbol{E})^{\mathrm{source}}_{\omega,\boldsymbol{k},\boldsymbol{p}}$ and $(\delta \boldsymbol{E}\ \delta \boldsymbol{E})^{\mathrm{source}}_{\omega,\boldsymbol{k}}$. We therefore use the following relation, obtained from (33.6) :

$$\delta \boldsymbol{E}(\omega,\boldsymbol{k}) = -\frac{i\boldsymbol{k}}{k^2} \sum_a 4\pi\, e_a \int d\boldsymbol{p}\ \delta N_a(\omega,\boldsymbol{k},\boldsymbol{p}) \tag{34.12}$$

From (34.11), (34.12) we thus get :

$$(\delta N_a \delta \boldsymbol{E})^{\mathrm{source}}_{\omega,\boldsymbol{k},\boldsymbol{p}} = i\,\frac{\boldsymbol{k}}{k^2}\, 4\pi\, e_a\, 2\pi\, \delta(\omega - \boldsymbol{k}\cdot\boldsymbol{v})\, f_a(\boldsymbol{p}) , \tag{34.13}$$

$$(\delta \boldsymbol{E}\cdot\delta \boldsymbol{E})^{\mathrm{source}}_{\omega,\boldsymbol{k}} = \sum_a \frac{(4\pi)^2 e_a^2 n_a}{k^2} \int d\boldsymbol{p}\, 2\pi\, \delta(\omega - \boldsymbol{k}\cdot\boldsymbol{v})\, f_a(\boldsymbol{p}) . \tag{34.14}$$

We see that the spectral densities (34.11), (34.13) have a resonant character (they are different from zero only for $\omega = \boldsymbol{k}\cdot\boldsymbol{v}$). The width of the resonance is of order $\Delta \sim \tau_\phi^{-1}$ and goes to zero as $\Delta \to 0$.

From (34.11) - (34.14) we may obtain, by integration over ω, the spatial spectral densities; for instance :

$$(\delta N_a \delta N_b)^{\mathrm{source}}_{\boldsymbol{k},\boldsymbol{p},\boldsymbol{p}'} = (2\pi)^{-1} \int d\omega\, (\delta N_a \delta N_b)^{\mathrm{source}}_{\omega,\boldsymbol{k},\boldsymbol{p},\boldsymbol{p}'}$$

$$= n_a \delta_{ab}\, \delta(\boldsymbol{p}-\boldsymbol{p}')\, f_a(\boldsymbol{p}) \tag{34.15}$$

If this expression is substituted into (34.6), one finds upon integration over $\boldsymbol{k}$ an expression which is identical to the initial condition (33.11).

35. SPECTRAL DENSITIES OF FLUCTUATIONS IN THE IDEAL PLASMA

We introduce a dissipative term $\Delta(\delta N_a - \delta N_a^{\mathrm{source}})$ into (33.24), as in Eq. (34.1) :

$$\left(\frac{\partial}{\partial t} + \boldsymbol{v}\cdot\frac{\partial}{\partial \boldsymbol{r}} + \Delta\right)(\delta N_a - \delta N_a^{\mathrm{source}}) = -e_a\, \delta \boldsymbol{E}\cdot\frac{\partial n_a f_a}{\partial \boldsymbol{p}} \tag{35.1}$$

We now easily obtain the solution for the Fourier transform $\delta N_a(\omega, \boldsymbol{k}, \boldsymbol{p})$, in the approximation that f_a is taken as independent of $\boldsymbol{r}$ and t:

$$\delta N_a(\omega, \boldsymbol{k}, \boldsymbol{p}) = \delta N_a^{\text{source}}(\omega, \boldsymbol{k}, \boldsymbol{p}) - \frac{i e_a}{\omega - \boldsymbol{k}\cdot\boldsymbol{v} + i\Delta}\, \delta \boldsymbol{E}(\omega, \boldsymbol{k}) \cdot \frac{\delta n_a f_a}{\partial \boldsymbol{p}}. \tag{35.2}$$

Equation (35.2), combined with (34.12), form a set of inhomogeneous equations among which δN_a can be excluded. We thus obtain:

$$\varepsilon(\omega, \boldsymbol{k})\; \delta \boldsymbol{E}(\omega, \boldsymbol{k}) = \delta \boldsymbol{E}^{\text{source}}(\omega, \boldsymbol{k}) \tag{35.3}$$

where the Fourier transform of the source of the fluctuations is

$$\delta \boldsymbol{E}^{\text{source}}(\omega, \boldsymbol{k}) = -\frac{i\boldsymbol{k}}{k^2} \sum_a 4\pi e_a \int d\boldsymbol{p}\; \delta N_a^{\text{source}}(\omega, \boldsymbol{k}, \boldsymbol{p}) \tag{35.4}$$

and the dielectric constant is

$$\varepsilon(\omega, \boldsymbol{k}) = 1 + \sum_a \frac{4\pi e_a^2 n_a}{k^2} \int d\boldsymbol{p}\, \frac{\boldsymbol{k}\cdot(\partial f_a/\partial \boldsymbol{p})}{\omega - \boldsymbol{k}\cdot\boldsymbol{v} + i\Delta}\,. \tag{35.5}$$

Equation (35.5) has the same form as (31.22) defining the dielectric constant which relates the average fields $\delta\boldsymbol{D}$ and $\delta\boldsymbol{E}$ in the approximation of the first moments. In (31.22) f_a^0 is a particular solution of the equations for f_a, $\boldsymbol{E}$, obtained by neglecting completely the correlations. In (35.5), f_a is a function obeying the kinetic equation, which includes the collisions. Equations (35.5), (31.22) are both valid for values of ω, $\boldsymbol{k}$ such that $\omega \gg \nu_{\text{rel}}$, $k \gg l_{\text{rel}}^{-1}$.

From (35.3) and (34.14) we obtain the space-time spectral density of the electric field fluctuations, for a Coulomb plasma :

$$\begin{aligned}(\delta\boldsymbol{E}\cdot\delta\boldsymbol{E})_{\omega,\boldsymbol{k}} &= \frac{(\delta\boldsymbol{E}\cdot\delta\boldsymbol{E})^{\text{source}}_{\omega,\boldsymbol{k}}}{|\varepsilon(\omega, \boldsymbol{k})|^2} \\ &= \frac{1}{|\varepsilon(\omega, \boldsymbol{k})|^2} \sum_a \frac{(4\pi)^2 e_a^2 n_a}{k^2} \int d\boldsymbol{p}\; 2\pi\, \delta(\omega - \boldsymbol{k}\cdot\boldsymbol{v})\, f_a.\end{aligned} \tag{35.6}$$

From this equation we obtain the spatial spectral density of the electric field fluctuations for the Coulomb plasma :

$$(\delta\boldsymbol{E}\cdot\delta\boldsymbol{E})_{\boldsymbol{k}} = \sum_a \frac{(4\pi)^2 e_a^2 n_a}{k^2} \int d\boldsymbol{p}\, \frac{f_a(\boldsymbol{p})}{|\varepsilon(\boldsymbol{k}\cdot\boldsymbol{v}, \boldsymbol{k})|^2}\,. \tag{35.7}$$

We now consider the spectral density $(\delta N_a\, \delta\boldsymbol{E})_{\omega,\boldsymbol{k},\boldsymbol{p}}$ which defines the collision integral. From (35.2), (35.3), recalling that $\delta\boldsymbol{E}$ is parallel to $\boldsymbol{k}$, we find

$$(\delta N_a\, \delta\boldsymbol{E})_{\omega,\boldsymbol{k},\boldsymbol{p}} = -\frac{i e_a n_a}{\omega - \boldsymbol{k}\cdot\boldsymbol{v} + i\Delta}\, \frac{\boldsymbol{k}}{k^2}\, (\delta\boldsymbol{E}\cdot\delta\boldsymbol{E})_{\omega,\boldsymbol{k}}\; \boldsymbol{k}\cdot\frac{\partial f_a}{\partial \boldsymbol{p}} + \frac{(\delta N_a\, \delta\boldsymbol{E})^{\text{source}}_{\omega,\boldsymbol{k},\boldsymbol{p}}}{\varepsilon^*(\omega, \boldsymbol{k})}\,. \tag{35.8}$$

With the result (34.13) we get

$$(\delta N_a\,\delta \boldsymbol{E})_{\omega,\boldsymbol{k},\boldsymbol{p}} = -\frac{i e_a n_a}{\omega - \boldsymbol{k}\cdot\boldsymbol{v} + i\Delta}\,\frac{\boldsymbol{k}}{k^2}\,(\delta \boldsymbol{E}\cdot\delta \boldsymbol{E})_{\omega,\boldsymbol{k}}\;\boldsymbol{k}\cdot\frac{\partial f_a}{\partial \boldsymbol{p}}$$
$$+\; i\,\frac{\boldsymbol{k}}{k^2}\,4\pi e_a n_a\,2\pi\delta(\omega - \boldsymbol{k}\cdot\boldsymbol{v})\,\frac{f_a(\boldsymbol{p})}{\varepsilon^*(\omega,\boldsymbol{k})}\,. \tag{35.9}$$

We separate out the real part, which defines the collision integral of the ideal plasma (35.5):

$$\mathrm{Re}\,(\delta N_a\,\delta \boldsymbol{E})_{\omega,\boldsymbol{k},\boldsymbol{p}} = -\,\pi e_a n_a\,\delta(\omega - \boldsymbol{k}\cdot\boldsymbol{v})\,\frac{\boldsymbol{k}}{k^2}\,(\delta \boldsymbol{E}\cdot\delta \boldsymbol{E})_{\omega,\boldsymbol{k}}\;\boldsymbol{k}\cdot\frac{\partial f_a}{\partial \boldsymbol{p}}$$
$$-\,\frac{4\pi e_a n_a}{k^2}\,\boldsymbol{k}\,2\pi\,\delta(\omega - \boldsymbol{k}\cdot\boldsymbol{v})\,\frac{\mathrm{Im}\,\varepsilon(\omega,\boldsymbol{k})}{|\varepsilon(\omega,\boldsymbol{k})|^2}\,f_a(\boldsymbol{p})\,. \tag{35.10}$$

In a similar way, we obtain an expression for the more general spectral density of the particle number fluctuations

$$(\delta N_a\,\delta N_b)_{\omega,\boldsymbol{k},\boldsymbol{p},\boldsymbol{p}'} = 2\pi\delta_{ab}\,\delta(\omega - \boldsymbol{k}\cdot\boldsymbol{v})\,\delta(\boldsymbol{p}-\boldsymbol{p}')\,n_a f_a$$
$$+\,\frac{e_a n_a e_b n_b(\delta \boldsymbol{E}\cdot\delta \boldsymbol{E})_{\omega,\boldsymbol{k}}}{k^2(\omega - \boldsymbol{k}\cdot\boldsymbol{v} + i\Delta)(\omega - \boldsymbol{k}\cdot\boldsymbol{v}' - i\Delta)}\left(\boldsymbol{k}\cdot\frac{\partial f_a}{\partial \boldsymbol{p}}\right)\left(\boldsymbol{k}\cdot\frac{\partial f_b}{\partial \boldsymbol{p}'}\right)$$
$$-\,\frac{4\pi e_a n_a e_b n_b}{k^2}\left[\frac{2\pi\delta(\omega - \boldsymbol{k}\cdot\boldsymbol{v}')}{\omega - \boldsymbol{k}\cdot\boldsymbol{v} + i\Delta}\,\frac{1}{\varepsilon(\omega,\boldsymbol{k})}\left(\boldsymbol{k}\cdot\frac{\partial f_a}{\partial \boldsymbol{p}}\right) f_b\right.$$
$$\left.+\,\frac{2\pi\delta(\omega - \boldsymbol{k}\cdot\boldsymbol{v})}{\omega - \boldsymbol{k}\cdot\boldsymbol{v}' - i\Delta}\,\frac{1}{\varepsilon^*(\omega,\boldsymbol{k})}\right]\left(\boldsymbol{k}\cdot\frac{\partial f_b}{\partial \boldsymbol{p}'}\right) f_a \tag{35.11}$$

The functions $\varepsilon(\omega,\boldsymbol{k})$, $(\delta \boldsymbol{E}\cdot\delta \boldsymbol{E})_{\omega,\boldsymbol{k}}$ entering Eqs (35.9) – (35.11) are defined by (35.5), (35.6).

From these results we can obtain expressions for the spatial spectral densities of the fluctuations. We consider in particular the function $\mathrm{Re}\,(\delta N_a\,\delta E)_{\boldsymbol{k},\boldsymbol{p}}$ which defines the collision integral according to (33.5). We introduce (35.6) into (35.10) and integrate over ω:

$$\mathrm{Re}\,(\delta N_a\,\delta \boldsymbol{E})_{\boldsymbol{k},\boldsymbol{p}}$$
$$= -\,\frac{16\pi^3 e_a n_a}{k^4}\,\boldsymbol{k}\sum_b e_b^2 n_b \int d\boldsymbol{p}'\,\frac{\delta(\boldsymbol{k}\cdot\boldsymbol{v} - \boldsymbol{k}\cdot\boldsymbol{v}')}{|\varepsilon(\boldsymbol{k}\cdot\boldsymbol{v},\boldsymbol{k})|^2}\;\boldsymbol{k}\cdot\left[\frac{\partial f_a}{\partial \boldsymbol{p}} f_b - \frac{\partial f_b}{\partial \boldsymbol{p}'} f_a\right]. \tag{35.12}$$

The expressions for the spectral densities depend on $\boldsymbol{r}$ and on t through the functions f_a. The latter obey the kinetic equations which will be derived in section 37. Thus the expressions found for the spectral densities are clearly nonequilibrium quantities.

The kinetic equations describe the processes of relaxation towards equilibrium. In the absence of external forces, the latter is characterized by a Maxwell

distribution. If this distribution is used in the formulae for the spectral densities, we obtain the corresponding equilibrium expressions. These will be studied in the next section.

36. SPECTRAL DENSITIES OF THE FLUCTUATIONS IN THE EQUILIBRIUM PLASMA

We first consider Eq. (35.6) for the spectral density $(\delta \boldsymbol{E} \cdot \delta \boldsymbol{E})_{\omega, \boldsymbol{k}}$. In equilibrium we can write:

$$\delta(\omega - \boldsymbol{k} \cdot \boldsymbol{v})\, f_a = -\,\delta(\omega - \boldsymbol{k} \cdot \boldsymbol{v}) \left(\boldsymbol{k} \cdot \frac{\partial f_a}{\partial \boldsymbol{p}} \right) \frac{k_{\mathrm{B}} T}{\omega} \ . \tag{36.1}$$

Thus (35.6) can be written as

$$(\delta \boldsymbol{E} \cdot \delta \boldsymbol{E})_{\omega, \boldsymbol{k}} = -\frac{1}{|\varepsilon(\omega, \boldsymbol{k})|^2} \sum_a \frac{(4\pi)^2 e_a^2 n_a}{\omega k^2}\, 2\pi \int d\boldsymbol{p}\, \delta(\omega - \boldsymbol{k} \cdot \boldsymbol{v})\, \boldsymbol{k} \cdot \frac{\partial f_a}{\partial \boldsymbol{p}} \ . \tag{36.2}$$

However, from (35.5), we have

$$\operatorname{Im} \varepsilon(\omega, \boldsymbol{k}) = -\sum_a \frac{4\pi^2 e_a^2 n_a}{k^2} \int d\boldsymbol{p}\, \delta(\omega - \boldsymbol{k} \cdot \boldsymbol{v})\, \boldsymbol{k} \cdot \frac{\partial f_a}{\partial \boldsymbol{p}} \tag{36.3}$$

Hence

$$(\delta \boldsymbol{E} \cdot \delta \boldsymbol{E})_{\omega, \boldsymbol{k}} = \frac{8\pi}{\omega} \frac{\operatorname{Im} \varepsilon(\omega, \boldsymbol{k})}{|\varepsilon(\omega, \boldsymbol{k})|^2}\, k_{\mathrm{B}} T \ . \tag{36.4}$$

Comparing (36.4) and (36.5) we also find the expression of the spectral density of the source fluctuations in equilibrium:

$$(\delta \boldsymbol{E} \cdot \delta \boldsymbol{E})^{\text{source}}_{\omega, \boldsymbol{k}} = \frac{8\pi}{\omega} \operatorname{Im} \varepsilon(\omega, \boldsymbol{k})\, k_{\mathrm{B}} T \tag{36.5}$$

Using the formula which relates the dielectric constant to the conductivity

$$\varepsilon(\omega, \boldsymbol{k}) = 1 + \frac{4\pi i}{\omega} \sigma(\omega, k) \ , \tag{36.6}$$

we also have

$$(\delta \boldsymbol{E} \cdot \delta \boldsymbol{E})^{\text{source}}_{\omega, \boldsymbol{k}} = \frac{32\pi^2}{\omega^2} \operatorname{Re} \sigma(\omega, \boldsymbol{k})\, k_{\mathrm{B}} T \ . \tag{36.7}$$

This formula is analogous to the well-known Nyquist formula for the spectral density of the electromotive force $\mathcal{E}$, equivalent to the thermal motion:

$$(\mathcal{E}^2)_\omega = 2R(\omega)\, k_{\mathrm{B}} T \tag{36.8}$$

where $R(\omega)$ is the ohmic resistance of the oscillating circuit.

From (36.7) we may obtain the formulae for the source fluctuations of charges and currents. From Poisson's equation we obtain

$$\delta q^{\text{source}}(\omega, \boldsymbol{k}) = \frac{i}{4\pi}\, \boldsymbol{k} \cdot \delta \boldsymbol{E}^{\text{source}} \tag{36.9}$$

From (36.9) and (36.7) we obtain the spectral density of the source fluctuations of charges :

$$(\delta q^2)^{\text{source}}_{\omega, \boldsymbol{k}} = \frac{2k^2}{\omega^2} \operatorname{Re} \sigma(\omega, \boldsymbol{k})\, k_{\mathrm{B}}T = \frac{k^2}{2\pi\omega} \operatorname{Im} \varepsilon(\omega, \boldsymbol{k})\, k_{\mathrm{B}}T \,. \tag{36.10}$$

From the charge conservation equation follows

$$\omega\, \delta q^{\text{source}} = \boldsymbol{k} \cdot \delta \boldsymbol{j}^{\text{source}} \quad \text{or} \quad 4\pi\, \delta \boldsymbol{j}^{\text{ source}} = i\omega\, \delta \boldsymbol{E}^{\text{ source}} \tag{36.11}$$

and thus

$$(\delta \boldsymbol{j} \cdot \delta \boldsymbol{j})^{\text{source}}_{\omega, \boldsymbol{k}} = 2 \operatorname{Re} \sigma(\omega, \boldsymbol{k})\, k_{\mathrm{B}}T \,. \tag{36.12}$$

We now consider the spatial density of field fluctuations in equilibrium. From (36.4) we find

$$\begin{aligned}(\delta \boldsymbol{E} \cdot \delta \boldsymbol{E})_{\boldsymbol{k}} &= \frac{1}{2\pi} k_{\mathrm{B}}T \int d\omega\, \frac{8\pi}{\omega} \frac{\operatorname{Im} \varepsilon(\omega, \boldsymbol{k})}{|\varepsilon(\omega, \boldsymbol{k})|^2} \\ &= \frac{4\pi}{2\pi i} \int_{-\infty}^{\infty} d\omega\, \tfrac{1}{2} k_{\mathrm{B}}T \left[\frac{1}{\omega - i\Delta} + \frac{1}{\omega + i\Delta}\right] \left[f^-(\omega, \boldsymbol{k}) - f^+(\omega, \boldsymbol{k}) \right]\end{aligned} \tag{36.13}$$

where

$$f^+ = 1 - \frac{1}{\varepsilon(\omega, \boldsymbol{k})} \,, \quad f^- = 1 - \frac{1}{\varepsilon^*(\omega, \boldsymbol{k})} \tag{36.14}$$

f^+ and f^- are analytic functions, in the upper and in the lower half-plane, respectively. Using the Cauchy formula for $\lim\limits_{\omega \to \infty} f(\omega) = 0$,

$$-\frac{1}{2\pi i} \int d\omega' \, \frac{f^{\pm}(\omega')}{\omega - \omega' \pm i\Delta} = \pm f^{\pm}(\omega)$$

$$-\frac{1}{2\pi i} \int d\omega' \, \frac{f^{\mp}(\omega')}{\omega - \omega' \pm i\Delta} = 0 \tag{36.15}$$

we find

$$(\delta \boldsymbol{E} \cdot \delta \boldsymbol{E})_{\boldsymbol{k}} = 4\pi \left[1 - \frac{1}{\varepsilon(0, \boldsymbol{k})} \right] k_{\mathrm{B}}T \,. \tag{36.16}$$

From the expression of the dielectric constant (35.5) follows that, in equilibrium

$$\varepsilon(0, \boldsymbol{k}) = 1 + \frac{1}{r_{\mathrm{D}}^2 k^2} \,. \tag{36.17}$$

Combining with (36.16) we obtain the final form for the equilibrium spectral density of the field fluctuations

$$(\delta \boldsymbol{E} \cdot \delta \boldsymbol{E})_{\boldsymbol{k}} = \frac{4\pi\, k_{\mathrm{B}}T}{1 + r_{\mathrm{D}}^2 k^2} \tag{36.18}$$

Comparing Eqs (35.7) and (36.18) we find that, in equilibrium, the following relation holds:

$$\sum_a e_a^2 n_a \int dp \; \frac{f_a}{|\varepsilon(\boldsymbol{k}\cdot\boldsymbol{v},\boldsymbol{k})|^2} = \sum_a e_a^2 n_a \frac{r_D^2 k^2}{1 + r_D^2 k^2} \tag{36.19}$$

This result will be used in section 47.

From Eq. (35.11) we find the following expression for the spatial spectral density of fluctuation of the particle numbers in equilibrium:

$$(\delta N_a \, \delta N_b)_{\boldsymbol{k},\boldsymbol{p},\boldsymbol{p}'} = \delta_{ab}\,\delta(\boldsymbol{p}-\boldsymbol{p}')\, n_a f_a(\boldsymbol{p}) - \frac{e_a e_b n_a n_b}{\sum_c e_c^2 n_c\,(1 + r_D^2 k^2)} f_a(\boldsymbol{p}) f_b(\boldsymbol{p}') \tag{36.20}$$

where f_a, f_b are Maxwellian.

Noting that

$$(\delta \boldsymbol{E}\cdot\delta \boldsymbol{E})_{\boldsymbol{k}} = \sum_a \sum_b \frac{(4\pi)^2 e_a e_b}{k^2} \int d\boldsymbol{p}\, d\boldsymbol{p}'\, (\delta N_a\, \delta N_b)_{\boldsymbol{k},\boldsymbol{p},\boldsymbol{p}'} \tag{36.21}$$

we see that (36.18) follows directly from (36.20).

Comparing (36.20) and (26.11) we get an expression for the Fourier transform of the correlation function g_{ab} in equilibrium:

$$g_{ab}(\boldsymbol{k},\boldsymbol{p},\boldsymbol{p}') = -\frac{e_a e_b}{\sum_c e_c^2 n_c} \frac{f_a(\boldsymbol{p})\, f_b(\boldsymbol{p}')}{1 + r_D^2 k^2}. \tag{36.22}$$

From the integral formula

$$(2\pi)^{-3}\int d\boldsymbol{k}\; e^{i\boldsymbol{k}\cdot\boldsymbol{r}} \frac{1}{1 + r_D^2 k^2} = \frac{1}{4\pi r_D^2} \frac{e^{-r/r_D}}{r} \tag{36.23}$$

we find the equilibrium correlation function

$$g_{ab}(\boldsymbol{r},\boldsymbol{p},\boldsymbol{p}') = -\frac{e_a e_b}{k_B T} \frac{e^{-r/r_D}}{r} f_a(\boldsymbol{p})\, f_b(\boldsymbol{p}') \tag{36.24}$$

Thus, the Debye radius plays the role of the correlation length: for $r \gg r_D$ the correlation function vanishes.

The two-particle function f_{ab} is given by

$$f_{ab} = f_a f_b + g_{ab} \tag{36.25}$$

and thus

$$f_{ab} = \left(1 - \frac{e_a e_b}{k_B T} \frac{e^{-r/r_D}}{r}\right) f_a f_b \tag{36.26}$$

For $r \gg r_D$, $g_{ab} \to 0$ and $f_{ab} \longrightarrow f_a f_b$.

From (36.24) we see that, for charges of the same sign, the correlation

function is negative $(f_{ab} < f_a f_b)$, whereas for charges of opposite sign it is positive $(f_{ab} > f_a f_b)$.

For short distances, $r \ll r_D$, Eq. (36.26) gives a condition under which the second term is small compared to the first. Thus, (36.26) is valid only for

$$r \gg \frac{|e_a e_b|}{k_B T} , \tag{36.27}$$

i.e., for distances where the potential energy is much smaller than the kinetic energy. For the Fourier transform, this implies that (36.22) is valid for wave vectors such that

$$k \ll \frac{k_B T}{|e_a e_b|} . \tag{36.28}$$

This condition will be used in the derivation of the collision integral I_a .

From (36.3) follows that, in equilibrium, the imaginary part of the dielectric constant is

$$\operatorname{Im} \varepsilon(\omega, \boldsymbol{k}) = \pi \sum_a \frac{4\pi e_a^2 n_a}{k^2 k_B T} \frac{\omega}{k} \left(\frac{m_a}{2\pi k_B T}\right)^{\frac{1}{2}} \exp\left(-\frac{m_a \omega^2}{2 k_B T k^2}\right) \tag{36.29}$$

Thus, for $\omega \neq 0$, $\operatorname{Im} \varepsilon(\omega, \boldsymbol{k}) \longrightarrow 0$ as $k \to 0$. Using the equation

$$\lim_{\Delta \to 0} \frac{\Delta}{\phi^2(x) + \Delta^2} = \pi \delta(\phi(x)) , \quad \Delta \to 0 \tag{36.30}$$

we find from (36.29), for small $\operatorname{Im} \varepsilon$:

$$\frac{\omega \operatorname{Im} \varepsilon(\omega, \boldsymbol{k})}{|\omega \varepsilon(\omega, \boldsymbol{k})|^2} = \pi \delta[\omega \operatorname{Re} \varepsilon(\omega, \boldsymbol{k})] \; k_B T . \tag{36.31}$$

Thus, for small k, Eq. (36.4) for the space-time spectral density of field fluctuations reduces to

$$(\delta \boldsymbol{E} \cdot \delta \boldsymbol{E})_{\omega, \boldsymbol{k}} = 8\pi^2 \delta[\omega \operatorname{Re} \varepsilon(\omega, \boldsymbol{k})] k_B T . \tag{36.32}$$

Thus, for small k, when $\operatorname{Im} \varepsilon$ is small (region of transparency), the spectral density is different from zero only for values of $\omega, \boldsymbol{k}$ related by the dispersion relation

$$\operatorname{Re} \varepsilon(\omega, \boldsymbol{k}) = 0 . \tag{36.33}$$

In zeroth approximation in k, we find from (35.5)

$$\operatorname{Re} \varepsilon(\omega, \boldsymbol{k}) = 1 - \frac{\omega_L^2}{\omega^2} \tag{36.34}$$

We make use of the equation

$$\delta[\phi(x)] = \sum_s \frac{\delta(x - x_s)}{|\phi'(x_s)|} \tag{36.35}$$

where x_s are the simple zeros of ϕ. We find from (36.32), (36,34), (36.35):

$$(\delta E \cdot \delta E)_{\omega, \boldsymbol{k}} = 4\pi^2 \left[\delta(\omega - \omega_L) + \delta(\omega + \omega_L)\right] k_{\mathrm{B}} T \tag{36.36}$$

Thus, the spectral density differs from zero only for $\omega = \pm\,\omega_L$ and is independent of $\boldsymbol{k}$. The spatial spectral density following from (36.36) is:

$$(\delta E \cdot \delta E)_{\boldsymbol{k}} = 4\pi\, k_{\mathrm{B}} T \tag{36.37}$$

which agrees with the limit of (36.18) as $k \to 0$.

More detailed investigations of the theory of the equilibrium properties of plasmas can be found in references [4, 8, 46, 47].

In sections 33 - 36 we developed the theory of short-range fluctuations for which $\tau_{\mathbf{cor}} \leqslant \omega_L^{-1}$, $r_{\mathbf{cor}} \leqslant r_{\mathbf{D}}$. We shall see that these fluctuations define the collision integral in the kinetic equations of plasmas. Besides the short-range fluctuations, an important role can be played in non-equilibrium plasma by long-range fluctuations: these can define, for instance, the turbulent state of the plasma. They also contribute to the dissipative state of the plasma. They may thus contribute important terms to the collision integrals and to the kinetic coefficients (see Chapter 11).

We shall see in Chapter 11 that the spectral density of the long-range fluctuations depend themselves on the form of that part of the collision integral, which is defined by the short-range fluctuations. On the contrary, the spectral densities of the short-range fluctuations are independent of the collision integral, as we just saw.

In a collisionless plasma, where the characteristic scales T, L are such that $T \ll \tau_{\mathbf{rel}}$, $L \ll l_{\mathbf{rel}}$, the short-range fluctuations, clearly, play the dominant role. In that case the spectral densities can significantly depend on the boundary conditions. The calculation of the spectral densities, taking the boundary conditions into account, was studied in the works of Ichimaru and Yakimenko [51] (see also [52, 53]).

37. THE BALESCU-LENARD KINETIC EQUATION

The collision integral is obtained by substituting (35.12) into (33.5)

$$I_a(\boldsymbol{r}, \boldsymbol{p}, t) = \sum_b 2\, e_a^2\, e_b^2\, n_b \frac{\partial}{\partial p_i} \int d\boldsymbol{k}\; d\boldsymbol{p}'\, \frac{k_i\, k_j}{k^4}\, \frac{\delta(\boldsymbol{k}\cdot\boldsymbol{v} - \boldsymbol{k}\cdot\boldsymbol{v}')}{|\varepsilon(\boldsymbol{k}\cdot\boldsymbol{v}, \boldsymbol{k})|^2}$$
$$\frac{\partial f_a(\boldsymbol{r}, \boldsymbol{p}, t)}{\partial p_j} f_b(\boldsymbol{r}, \boldsymbol{p}', t) - \frac{\partial f_b(\boldsymbol{r}, \boldsymbol{p}', t)}{\partial p_j'} f_a(\boldsymbol{r}, \boldsymbol{p}, t) \tag{37.1}$$

This expression was first obtained in the works of Balescu and of Lenard [5, 6] and is therefore called the Balescu-Lenard collision integral.

In Eq. (37.1) the variables $\boldsymbol{r}, t$ enter as parameters. As the spatial variation of f_a, f_b is not considered in the derivation of the collision integral, these two functions are evaluated at the same position.

We consider now alternative forms of this equation. From the first expression (33.5) and (35.10) we obtain

$$I_a(x,t) = \frac{e_a^2}{16\pi^3} \frac{\partial}{\partial p_i} \int d\omega \, d\boldsymbol{k} \, \delta(\omega - \boldsymbol{k}\cdot\boldsymbol{v}) \left\{ \left(\delta E_i \, \delta E_j\right)_{\omega,\boldsymbol{k}} \frac{\partial f_a}{\partial p_j} + \frac{8\pi \operatorname{Im}\varepsilon(\omega,\boldsymbol{k})}{|\varepsilon(\omega,\boldsymbol{k})|^2} \frac{k_i}{k^2} f_a \right\} \tag{37.2}$$

where

$$\left(\delta E_i \, \delta E_j\right)_{\omega,\boldsymbol{k}} = \frac{k_i k_j}{k^2} \left(\delta \boldsymbol{E} \cdot \delta \boldsymbol{E}\right)_{\omega,\boldsymbol{k}} . \tag{37.3}$$

Thus, we expressed the collision integral in terms of the spectral density of the field fluctuations and of the dielectric constant. These functions are related to f_a by (35.6), (35.5).

The representation (37.2) of the collision integral is quite convenient for the study of the relativistic kinetic equation (see section 39) and for the non-ideal effects (Chapter 9).

The collision integral can be written as a diffusion operator in momentum space :

$$I_a = \frac{\partial}{\partial p_i} D_{ij}^{(a)} \frac{\partial f_a}{\partial p_j} + \frac{\partial}{\partial p_i} \left(A_i^{(a)} f_a\right) . \tag{37.4}$$

The coefficients of diffusion and of friction are obtained for instance from (37.2) :

$$D_{ij}^{(a)}(\boldsymbol{p}) = \frac{e_a^2}{16\pi^3} \int d\omega \, d\boldsymbol{k} \, \delta(\omega - \boldsymbol{k}\cdot\boldsymbol{v}) \left(\delta E_i \, \delta E_j\right)_{\omega,\boldsymbol{k}} \tag{37.5}$$

$$A_i^{(a)}(\boldsymbol{p}) = \frac{e_a^2}{2\pi^2} \int d\omega \, d\boldsymbol{k} \; \delta(\omega - \boldsymbol{k}\cdot\boldsymbol{v}) \frac{k_i}{k^2} \frac{\operatorname{Im}\varepsilon(\omega,\boldsymbol{k})}{|\varepsilon(\omega,\boldsymbol{k})|^2} . \tag{37.6}$$

Equations (26.3) with the collision integral (37.1) or (37.2) or (36.4) are a closed set of kinetic equations for the plasma in the polarization approximation. The Balescu-Lenard collision integral in the form (37.1) was first obtained by solving the integral equations for the correlation functions g_{ab} (27.4) (see [5, 6, 8, 10]).

We now consider the general properties of the Balescu-Lenard collision integral.

We first show that on substituting the Maxwell distribution for f_a, the collision integral I_a vanishes. This follows from the simple property of the Maxwellian

$$\delta(\boldsymbol{k}\cdot\boldsymbol{v} - \boldsymbol{k}\cdot\boldsymbol{v}') \, \boldsymbol{k}\cdot\left\{\frac{\partial f_a}{\partial \boldsymbol{p}} f_b - \frac{\partial f_b}{\partial \boldsymbol{p}'} f_a\right\} = -\frac{(\boldsymbol{k}\cdot\boldsymbol{v} - \boldsymbol{k}\cdot\boldsymbol{v}')}{k_B T} \delta(\boldsymbol{k}\cdot\boldsymbol{v} - \boldsymbol{k}\cdot\boldsymbol{v}') \, f_a f_b$$

and from the definition of the δ-function

$$x\,\delta(x) = 0 \,. \tag{37.7}$$

Thus, the Maxwell distribution is a stationary solution of the kinetic equation.

Consider now the integrals

$$I(\boldsymbol{r}\,,t) = \sum_a n_a \int d\boldsymbol{p}\,\phi_a(\boldsymbol{p})\,I_a(x\,,t)\,, \tag{37.8}$$

where $\phi_a(\boldsymbol{p})$ is an arbitrary function. We substitute (37.1) into (37.8), integrate by parts on $\boldsymbol{p}$ and symmetrize with respect to $a,\boldsymbol{p}\,;\,b,\boldsymbol{p}'$:

$$I(\boldsymbol{r},t) = \sum_a \sum_b e_a^2 e_b^2 n_a n_b \int d\boldsymbol{k}\,d\boldsymbol{p}\,d\boldsymbol{p}' \left(\frac{\partial\phi_b}{\partial p_i'} - \frac{\partial\phi_a}{\partial p_i}\right)\frac{k_i k_j}{k^4}$$
$$\frac{\delta(\boldsymbol{k}\cdot\boldsymbol{v} - \boldsymbol{k}\cdot\boldsymbol{v}')}{|\varepsilon(\boldsymbol{k}\cdot\boldsymbol{v}\,,\boldsymbol{k})|^2}\left\{\frac{\partial f_a}{\partial p_j} f_b - \frac{\partial f_b}{\partial p_j'} f_a\right\}. \tag{37.9}$$

It follows that

$$I(\boldsymbol{r}\,,t) = 0 \quad \text{for} \quad \phi_a(\boldsymbol{p}) = 1\,,\boldsymbol{p}\,,\frac{p^2}{2m_a}\,. \tag{37.10}$$

The derivation of the first two properties is obvious; to derive the third one, one uses (37.7). The properties (37.10) of the ideal plasma are analogous to those of the ideal gases (11.5).

We now show that, taking into account the correlations which define the collision integral, the total entropy of the plasma:

$$S(t) = -k_b \sum_a n_a \int dx\, f_a \ln f_a \tag{37.11}$$

is growing and tends to a constant in equilibrium.

We substitute into (37.8) the function

$$\phi_a = -k_B \ln f_a \tag{37.12}$$

and introduce the vector

$$\boldsymbol{A} = \frac{\partial}{\partial\boldsymbol{p}} \ln f_a - \frac{\partial}{\partial\boldsymbol{p}'} \ln f_b\,. \tag{37.13}$$

Then (37.8) becomes

$$I(\boldsymbol{r},t) = k_B \sum_a \sum_b e_a^2 e_b^2 n_a n_b \int d\boldsymbol{k}\,d\boldsymbol{p}\,d\boldsymbol{p}'\,\frac{(\boldsymbol{k}\cdot\boldsymbol{A})^2}{k^4|\varepsilon(\boldsymbol{k}\cdot\boldsymbol{v}\,,\boldsymbol{k})|^2}$$
$$\delta(\boldsymbol{k}\cdot\boldsymbol{v} - \boldsymbol{k}\cdot\boldsymbol{v}')\, f_a f_b \geqslant 0 \tag{37.14}$$

because the integrand is positive. The value zero is attained when

$$\frac{\partial}{\partial\boldsymbol{p}} \ln f_a - \frac{\partial}{\partial\boldsymbol{p}'} \ln f_b = \alpha(\boldsymbol{p} - \boldsymbol{p}')\,, \tag{37.15}$$

where α is a constant, independent of the indices a, b. If this condition is satisfied, then not only $I(\boldsymbol{r}, t)$ but the collision integral itself vanishes.

The solution of Eq. (37.15) is

$$\ln f_a(\boldsymbol{p}) = \frac{1}{2}\alpha p^2 + \boldsymbol{\beta}\cdot\boldsymbol{p} + \gamma \tag{37.16}$$

hence $f_a(\boldsymbol{p})$ is the Maxwell distribution

$$f_a(\boldsymbol{p}) = \left(2\pi m_a k_B T\right)^{-\frac{3}{2}} \exp\left[-\frac{(\boldsymbol{p}-m_a u)^2}{2 m_a k_B T}\right]. \tag{37.17}$$

From (37.14), using the kinetic equation, we easily establish:

$$dS/dt \geqslant 0 . \tag{37.18}$$

The entropy reaches the maximum value when the distribution functions $f_a(x,t)$ attain the equilibrium, i.e. the Maxwell distributions for all components of the plasma, with the same T and $\boldsymbol{u}$.

We note that the integral over $\boldsymbol{k}$ in (37.1) diverges logarithmically at the upper limit. Indeed, for $k \to \infty$ $(k \gg r_D^{-1})$, $\varepsilon \to 1$ and the integrand behaves like k^{-1}.

The cause of this divergence lies in the fact that for small distances $(r \ll r_D)$ the polarization approximation becomes identical to the weak-coupling approximation. The formulae obtained here are therefore valid only under the conditions (36.27), (36.28). The limit of integration over k can be approximately defined by:

$$k_{\mathbf{max}} \sim \frac{k_B T}{|e_a e_b|} . \tag{37.19}$$

As the divergence at large k is logarithmic, the exact value of this cut-off has only a small influence on the result.

When the nonideal effects are taken into account, the additional contributions to the collision integral display a stronger divergence at small distances (large k). In the same time, the contribution of the small-distance interactions to the thermodynamic functions is important. Therefore, in the kinetic theory of nonideal plasmas, the contribution of the short-range interactions $(r \ll r_D)$ to the collision integral must be investigated more precisely (see sections 55, 56 in Chapter 9).

38. THE LANDAU KINETIC EQUATION

The Balescu-Lenard collision integral is quite complicated, as the distribution functions f_a enter not only explicitly (in the bracketed terms in (37.1)) but also through the dielectric constant, which defines the polarization of the medium. Because of this, an important role is played by simpler expressions of the collision integral, which however, retain some of the main features of the Balescu-Lenard equation.

An example of such a simpler collision integral is provided by the Landau equation. In order to derive it from the Balescu-Lenard equation, one proceeds as follows.

In Eq. (37.1) the polarization is defined by the dielectric constant $\varepsilon(\omega,\boldsymbol{k})$ at the frequency $\boldsymbol{k}\cdot\boldsymbol{v}$. One may say that (37.1) takes into account the dynamical polarization of the plasma. We shall now consider only the static polarization. We therefore replace $\varepsilon(\omega,\boldsymbol{k})$ in (37.1) by $\varepsilon(0,\boldsymbol{k})$, the latter being evaluated in equilibrium. Thus, from (36.17) :

$$\varepsilon(0,\boldsymbol{k}) = 1 + \frac{1}{r_{\mathrm{D}}^2 k^2} . \tag{38.1}$$

As the dielectric constant appears in (37:1) in the denominator, the main contribution to the k-integral comes from the region where $k > r_{\mathrm{D}}^{-1}$. We therefore set $\varepsilon = 1$ in the collision integral, but introduce the following limitations on the bounds of the k-integration :

$$r_{\mathrm{D}}^{-1} < k < k_{\max} = (k_{\mathrm{B}}T/|e_a e_b|) . \tag{38.2}$$

Then the collision integral takes the form :

$$\mathrm{I}_{\mathrm{a}} = \sum_{\mathrm{b}} \mathrm{n}_{\mathrm{b}} \frac{\partial}{\partial \mathrm{p}_{\mathrm{i}}} \int d\boldsymbol{p}' \, Q_{ij}^{(ab)} \left\{ \frac{\partial f_a}{\partial p_j} f_b - \frac{\partial f_b}{\partial p'_j} f_a \right\} \tag{38.3}$$

with

$$Q_{ij}^{(ab)} = 2\, e_a^2 e_b^2 \int \mathrm{d}\boldsymbol{k} \, \frac{k_i k_j}{k^4} \, \delta(\boldsymbol{k}\cdot\boldsymbol{v} - \boldsymbol{k}\cdot\boldsymbol{v}') \tag{38.4}$$

where the integration on the absolute value of k is done between the limits given in (38.2).

Performing the k-integration in (38.4) we find

$$Q_{ij}^{(ab)} = \frac{2\pi e_a^2 e_b^2}{|\boldsymbol{v}-\boldsymbol{v}'|^3} \, \ln \left(\frac{r_{\mathrm{D}} k_{\mathrm{B}} T}{|e_a e_b|} \right) \left\{ |\boldsymbol{v}-\boldsymbol{v}'| \, \delta_{ij} - (\boldsymbol{v}-\boldsymbol{v}')_i \, (\boldsymbol{v}-\boldsymbol{v}')_j \right\} \tag{38.5}$$

The equation for the distribution function with the collision integral (38.3), (38.4), is called the Landau kinetic equation. It was first derived [1] from the Boltzmann equation applied to a system of charged particles. However, we know that in a plasma there are many particles in a Debye sphere. Therefore, the use of the Boltzmann equation for the derivation of the Landau equation is not adequate. This can be seen from the fact that in obtaining (38.4) from the Boltzmann equation, there appears a logarithmic divergence at both ends. One must therefore make additional assumptions for a small $-k$ cut-off.

The use of the Landau collision integral considerably simplifies many calculations, e.g., the coefficients of viscosity, of heat conduction, of electrical conductivity, etc.

However, the use of the Landau equation instead of the Balescu-Lenard equation is not always possible. For large departures from equilibrium, when the contribution of the spectral densities in the region of transparency (small k) is important, the role of the dynamical polarization cannot be neglected, and the Balescu-Lenard collision integral must be used. Some examples of that kind are given in refs. [10, 30].

In conclusion, we now estimate the electron-electron and ion-ion collision frequencies, ν_{ee}, ν_{ii}. For an electron-ion plasma ($a = e, i$) the collision integrals can be written as follows:

$$I_e = I_{ee} + I_{ei} \quad , \quad I_i = I_{ii} + I_{ie} \tag{38.6}$$

where $I_{ee}(I_{ei})$ is the contribution of the electron-electron (electron-ion) collisions to the complete electron collision integral. Similarly, $I_{ii}(I_{ie})$ is the contribution of the ion-ion (ion-electron) collisions to the ion collision integral.

Equation (38.3) defines all four collision integrals. We first estimate the electron-electron collision frequency $\nu_{ee} \equiv \tau_{ee}^{-1}$. We set in (38.3), (38.5) for $I_{ee}: p \sim m_e v_T$ and obtain

$$I_{ee} \sim \nu_{ee} f_e = \frac{1}{\tau_{ee}} f_e \tag{38.7}$$

with

$$\nu_{ee} \sim \frac{e^4 n}{m^{\frac{1}{2}} (k_B T)^{\frac{3}{2}}} L \tag{38.8}$$

$L = \ln (r_D k_B T / |e_a e_b|)$ is the 'Coulomb logarithm'. Noting that $r_D = (k_B T / 8\pi e^2 n)^{\frac{1}{2}}$, $\omega_e = (4\pi e^2 n / m_e)^{\frac{1}{2}}$, $\mu = (n r_D^3)^{-1}$, we may also write for ν_{ee}:

$$\nu_{ee} \sim \omega_e \mu \ , \quad (l_{ee} \equiv v_T / \nu_{ee} \sim r_D / \mu) \ . \tag{38.9}$$

As the plasma parameter is small for a dilute plasma, $\mu \ll 1$, it follows that the collision frequency ν_{ee} is much smaller than the electron Langmuir frequency, and the mean free path is much larger than the Debye radius.

To estimate the quantity ν_{ii} we may use Eq. (38.8), changing $m_e \to m_i$ (we assume $T_e \approx T_i$, i.e., an isothermal plasma). Thus (for singly charged ions):

$$\nu_{ii} \sim \frac{e^4 n}{m_i^{\frac{1}{2}} (k_B T)^{\frac{3}{2}}} L \sim \left(\frac{m_e}{m_i}\right)^{\frac{1}{2}} \nu_{ee} \ll \nu_{ee} \ . \tag{38.10}$$

Using also Eq. (38.9) we obtain:

$$\nu_{ii} \sim \left(\frac{m_e}{m_i}\right)^{\frac{1}{2}} \omega_e \mu = \omega_i \mu \tag{38.11}$$

where $\omega_i = (4\pi e^2 n / m_i)^{\frac{1}{2}}$ is the ion Langmuir frequency.

The electron-ion collisions define various processes, such as the equalization of average velocities $\boldsymbol{u}_e, \boldsymbol{u}_i$, the equalization of temperatures T_e, T_i the

electrical conductivity.

The process of velocity equalization proceeds in a time scale of order τ_{ee}, i.e., its characteristic frequency is ν_{ee}. The same frequency also enters the expression of the electrical conductivity. The process of temperature equalization is defined by the electron-ion collision frequency $\nu_{ei}^{(T)}$. (The superscript T refers to the temperature equalization rate):

$$\nu_{ei}^{(T)} \sim \frac{m_e}{m_i} \nu_{ee} \sim \left(\frac{m_e}{m_i}\right) \nu_{ii} . \tag{38.12}$$

As $m_e \ll m_i$, we have

$$\nu_{ee} \gg \nu_{ii} \gg \nu_{ei}^{(T)} , \qquad \tau_{ee} \ll \tau_{ii} \ll \tau_{ei}^{(T)} . \tag{38.13}$$

Thus, there exist several stages in the relaxation processes in a plasma. In a time of order τ_{ee}, a local equilibrium distribution is established for the electrons, in a time of order $\tau_{ii} \gg \tau_{ee}$, the ion local equilibrium is reached. Thus, the ion and electron Maxwellians may have different temperatures T_i, T_e. The equalization of the temperatures is a much slower process, proceeding on a time scale $\tau_{ei}^{(T)} \gg \tau_{ee}, \tau_{ii}$

39. SPECTRAL DENSITIES OF FLUCTUATIONS IN A RELATIVISTIC PLASMA

For the calculation of the correlation of the fluctuations of phase densities δN_a and of electromagnetic fields $\delta\boldsymbol{E}, \delta\boldsymbol{B}$ we use Eqs (27.7), (26.20)–(26.23).

We first consider the case of a spatially homogeneous and isotropic plasma for which :

$$f_a = f_a(|\boldsymbol{p}|, t) . \tag{39.1}$$

In this case (see Eq. (27.7)) :

$$\frac{e_a}{c} (\boldsymbol{v} \times \delta\boldsymbol{B}) \cdot \frac{\partial}{\partial \boldsymbol{p}} f_a(|\boldsymbol{p}|, t) \equiv 0 . \tag{39.2}$$

Adding to (27.7) a dissipative term $(\delta N_a - \delta N_a^{\textbf{source}})$ we obtain an equation identical with (35.1). We may thus use Eq. (35.2) to obtain

$$\delta N_a(\omega, \boldsymbol{k}, \boldsymbol{p}) = \delta N_a^{\textbf{source}} - \frac{i e_a n_a}{\omega - \boldsymbol{k}\cdot\boldsymbol{v} + i\Delta} \, \delta\boldsymbol{E}(\omega, \boldsymbol{k}) \cdot \frac{\partial f_a}{\partial \boldsymbol{p}} . \tag{39.3}$$

The difference with respect to (35.2) is that now $\delta\boldsymbol{E} = \delta\boldsymbol{E}^{\parallel} + \delta\boldsymbol{E}^{\perp}$, i.e., that the electric field has both longitudinal and transverse components (with respect to the wavevector $\boldsymbol{k}$). From (26.20) – (26.23) we obtain the equations for the fluctuations $\delta E(\omega, \boldsymbol{k})$, $\delta\boldsymbol{B}(\omega, \boldsymbol{k})$:

$$i\boldsymbol{k} \times \delta\boldsymbol{B} = -\frac{i\omega}{c} \delta\boldsymbol{E} + \frac{4\pi}{c} \sum_a e_a \int d\boldsymbol{p} \, \boldsymbol{v} \, \delta N_a \tag{39.4}$$

$$i\boldsymbol{k}\times\delta\boldsymbol{E} = \frac{i\omega}{c}\,\delta\boldsymbol{B} \tag{39.5}$$

$$i\boldsymbol{k}\cdot\delta\boldsymbol{B} = 0 \tag{39.6}$$

$$i\boldsymbol{k}\cdot\delta\boldsymbol{E} = 4\pi\sum_a e_a\int d\boldsymbol{p}\,\delta N_a\,. \tag{39.7}$$

Because of its linearity, this set can be split into two sets. for the transverse fluctuations $\delta\boldsymbol{E}^{\perp}, \delta\boldsymbol{B}^{\perp}$, and for the longitudinal ones, $\delta\boldsymbol{E}^{\parallel}$, respectively. We therefore represent the vectors $\delta\boldsymbol{E}, \delta\boldsymbol{j}$ as follows :

$$\delta\boldsymbol{E} = \delta\boldsymbol{E}^{\parallel} + \delta\boldsymbol{E}^{\perp}\,, \qquad \boldsymbol{k}\times\delta\boldsymbol{E}^{\parallel} = 0\,, \qquad \boldsymbol{k}\cdot\delta\boldsymbol{E}^{\perp} = 0 \tag{39.8}$$

$$\delta\boldsymbol{j} = \delta\boldsymbol{j}^{\parallel} + \delta\boldsymbol{j}^{\perp}\,, \qquad \boldsymbol{k}\times\delta\boldsymbol{j}^{\parallel} = 0\,, \qquad \boldsymbol{k}\cdot\delta\boldsymbol{j}^{\perp} = 0\,. \tag{39.9}$$

For the longitudinal part we find from (39.4) – (39.7) :

$$i\boldsymbol{k}\times\delta\boldsymbol{E} = 0\,, \qquad i\boldsymbol{k}\cdot\delta\boldsymbol{E}^{\parallel} = 4\pi\sum e_a\int d\boldsymbol{p}\,\delta N_a\,, \tag{39.10}$$

or, using the equation for the total current :

$$i\boldsymbol{k}\times\delta\boldsymbol{E}^{\parallel} = 0\,, \quad -i\omega\delta\boldsymbol{E}^{\parallel} = -4\pi\delta\boldsymbol{j}^{\parallel} = -4\pi\sum_a e_a\int d\boldsymbol{p}\,\boldsymbol{k}\,\frac{\boldsymbol{k}\cdot\boldsymbol{v}}{k^2}\,\delta N_a\,. \tag{39.11}$$

The equations for the transverse fields become :

$$i\boldsymbol{k}\times\delta\boldsymbol{B} = -\frac{i\omega}{c}\,\delta\boldsymbol{E}^{\perp} + \frac{4\pi}{c}\sum_a e_a\int d\boldsymbol{p}\,\frac{(\boldsymbol{k}\times\boldsymbol{v})\times\boldsymbol{k}}{k^2}\,\delta N_a \tag{39.12}$$

$$i\boldsymbol{k}\times\delta\boldsymbol{E}^{\perp} = \frac{i\omega}{c}\,\delta\boldsymbol{B}\,, \quad i\boldsymbol{k}\cdot\delta\boldsymbol{B} = 0\,, \qquad i\boldsymbol{k}\cdot\delta\boldsymbol{E} = 0\,. \tag{39.13}$$

In Eq. (39.12) we used the following identity :

$$\boldsymbol{A} = \frac{\boldsymbol{k}(\boldsymbol{k}\cdot\boldsymbol{A})}{k^2} + \boldsymbol{A} - \frac{\boldsymbol{k}(\boldsymbol{k}\cdot\boldsymbol{A})}{k^2} = \frac{\boldsymbol{k}(\boldsymbol{k}\cdot\boldsymbol{A})}{k^2} + \frac{(\boldsymbol{k}\times\boldsymbol{A})\times\boldsymbol{k}}{k^2} = \boldsymbol{A}^{\parallel} + \boldsymbol{A}^{\perp}\,. \tag{39.14}$$

Eliminating $\delta\boldsymbol{B}$ from (39.12), (39.13) we obtain :

$$\begin{aligned}(\omega^2 - c^2k^2)\,\delta\boldsymbol{E}^{\perp} &= -4\pi i\omega\,\delta\boldsymbol{j}^{\perp}\\ &= -4\pi i\omega\sum_a d\boldsymbol{p}\,\frac{(\boldsymbol{k}\times\boldsymbol{v})\times\boldsymbol{k}}{k^2}\,\delta N_a\end{aligned} \tag{39.15}$$

$$\delta\boldsymbol{B} = \frac{c}{\omega}\,(\boldsymbol{k}\times\delta\boldsymbol{E}^{\perp}) \tag{39.16}$$

From (39.11), (39.15) follows that the fluctuations of the transverse and longitudinal fields are expressed in terms of $\delta\boldsymbol{j}^{\perp}, \delta\boldsymbol{j}^{\parallel}$, respectively. We now express the latter in terms of the source fluctuations. We substitute (39.3) into the expression of the current :

$$\delta \boldsymbol{j}(\omega, \boldsymbol{k}) = \sum_a e_a \int d\boldsymbol{p}\ \boldsymbol{v}\ \delta N_a(\omega, \boldsymbol{k}, \boldsymbol{p}) \tag{39.17}$$

and rewrite the expression obtained in this way as

$$\delta j_i = \sigma_{ij}\,\delta E_j + j_i^{\text{source}} \tag{39.18}$$

where

$$\sigma_{ij} = -i \sum_a e_a^2 n_a \int d\boldsymbol{p}\ \frac{v_i(\partial f_a/\partial p_j)}{\omega - \boldsymbol{k}\cdot\boldsymbol{v} + i\Delta} \tag{39.19}$$

is the tensor of electrical conductivity, and

$$\delta \boldsymbol{j}^{\text{ source}} = \sum_a e_a \int d\boldsymbol{p}\ \boldsymbol{v}\ \delta N_a^{\text{source}}(\omega, \boldsymbol{k}, \boldsymbol{p}) \tag{39.20}$$

is the Fourier transform of the source of the current fluctuations.

Equation (39.19) has the same form as (31.29) for the conductivity tensor of a collisionless plasma. Therefore (see (31.32)) we introduce the dielectric tensor by:

$$\varepsilon_{ij} = \delta_{ij} + \frac{4\pi i}{\omega}\sigma_{ij} = \delta_{ij} + \sum_a \frac{4\pi e_a^2 n_a}{\omega} \int d\boldsymbol{p}\ \frac{v_i(\partial f_a/\partial p_j)}{\omega - \boldsymbol{k}\cdot\boldsymbol{v} + i\Delta} \tag{39.21}$$

The difference of these formulae with (31.29), (31.33) is in the fact that here f_a is defined by a kinetic equation, taking into account the collisions.

For an isotropic plasma, the tensors $\sigma_{ij}, \varepsilon_{ij}$ can be represented as follows:

$$\begin{aligned} \sigma_{ij} &= \frac{k_i k_j}{k^2}\sigma^{\parallel} + \left(\delta_{ij} - \frac{k_i k_j}{k^2}\right)\sigma^{\perp} \\ \varepsilon_{ij} &= \frac{k_i k_j}{k^2}\varepsilon^{\parallel} + \left(\delta_{ij} - \frac{k_i k_j}{k^2}\right)\varepsilon^{\perp} \end{aligned} \tag{39.22}$$

from which we find

$$\begin{aligned} \varepsilon^{\parallel} &= \frac{k_i k_j}{k^2}\varepsilon_{ij} \\ \varepsilon^{\perp} &= \frac{1}{2}\left(\delta_{ij} - \frac{k_i k_j}{k^2}\right)\varepsilon_{ij} \end{aligned} \tag{39.23}$$

and similar relations for $\sigma^{\parallel}, \sigma^{\perp}$.

From (39.19), (39.21), (39.23) we find

$$\varepsilon^{\parallel}(\omega, \boldsymbol{k}) = 1 + \sum_a \frac{4\pi e_a^2 n_a}{k^2\omega} \int d\boldsymbol{p}\ \frac{(\boldsymbol{k}\cdot\boldsymbol{v})\,\boldsymbol{k}\cdot(\partial f_a/\partial \boldsymbol{p})}{\omega - \boldsymbol{k}\cdot\boldsymbol{v} + i\Delta} \tag{39.24}$$

$$\varepsilon^{\perp}(\omega, \boldsymbol{k}) = 1 + \sum_a \frac{2\pi e_a^2 n_a}{k^2\omega} \int d\boldsymbol{p}\ \frac{[(\boldsymbol{k}\times\boldsymbol{v})\times\boldsymbol{k}]\cdot(\partial f_a/\partial \boldsymbol{p})}{\omega - \boldsymbol{k}\cdot\boldsymbol{v} + i\Delta} \tag{39.25}$$

Equation (39.24) differs only in the writing from (35.5).

Coming back to (39.18) and substituting into it Eq. (39.22) for σ_{ij}, we obtain

$$\delta \boldsymbol{j}^{\parallel} = \sigma^{\parallel}\, \delta \boldsymbol{E}^{\parallel} + \delta \boldsymbol{j}^{\parallel\,\mathrm{source}}$$
$$\delta \boldsymbol{j}^{\perp} = \sigma^{\perp}\, \delta \boldsymbol{E}^{\perp} + \delta \boldsymbol{j}^{\perp\,\mathrm{source}} \tag{39.26}$$

where

$$\delta \boldsymbol{j}^{\parallel\,\mathrm{source}} = \frac{\boldsymbol{k}(\boldsymbol{k}\cdot\delta \boldsymbol{j}^{\mathrm{source}})}{k^2}$$
$$\delta \boldsymbol{j}^{\perp\,\mathrm{source}} = \frac{(\boldsymbol{k}\times\delta \boldsymbol{j}^{\mathrm{source}})\times\boldsymbol{k}}{k^2} \tag{39.27}$$

We can now derive the expressions for the components of $\delta\boldsymbol{E}$ by substituting (39.26) into (39.10) and (39.15):

$$\delta \boldsymbol{E}^{\parallel} = \frac{\delta \boldsymbol{E}^{\parallel\,\mathrm{source}}}{\varepsilon^{\parallel}(\omega\,,\,\boldsymbol{k})}\,, \qquad \delta \boldsymbol{E}^{\perp} = \frac{\omega^2\,\delta \boldsymbol{E}^{\perp\,\mathrm{source}}}{\omega^2\,\varepsilon^{\perp} - c^2 k^2} \tag{39.28}$$

where we introduced the notations

$$\delta \boldsymbol{E}^{\parallel\,\mathrm{source}} = -\frac{4\pi i}{\omega}\,\delta \boldsymbol{j}^{\parallel\,\mathrm{source}}\,, \qquad \delta \boldsymbol{E}^{\perp\,\mathrm{source}} = -\frac{4\pi i}{\omega}\,\delta \boldsymbol{j}^{\perp\,\mathrm{source}} \tag{39.29}$$

for the sources of fluctuations of the longitudinal and transverse fields.

From (39.28) we find:

$$(\delta \boldsymbol{E}^{\parallel}\cdot\delta \boldsymbol{E}^{\parallel})_{\omega,\boldsymbol{k}} = \frac{1}{|\varepsilon^{\parallel}(\omega,\boldsymbol{k})|^2}\sum_a \frac{(4\pi)^2 e_a^2 n_a}{k^2}\int d\boldsymbol{p}\, 2\pi\,\delta(\omega-\boldsymbol{k}\cdot\boldsymbol{v}) f_a \tag{39.30}$$

$$(\delta \boldsymbol{E}^{\perp}\cdot\delta \boldsymbol{E}^{\perp})_{\omega,\boldsymbol{k}} = \frac{1}{|\omega^2\varepsilon^{\perp}(\omega,\boldsymbol{k}) - c^2 k^2|^2}\sum_a \frac{(4\pi)^2 e_a^2 n_a}{k^2}\,\omega^2\int d\boldsymbol{p}\, 2\pi\,\delta(\omega-\boldsymbol{k}\cdot\boldsymbol{v})\,|\boldsymbol{k}\times\boldsymbol{v}|^2 f_a \tag{39.31}$$

where we used the following expressions for the spectral densities of the source fluctuations of the fields:

$$(\delta \boldsymbol{E}^{\parallel}\cdot\delta \boldsymbol{E}^{\parallel})^{\mathrm{source}}_{\omega,\boldsymbol{k}} = \sum_a \frac{(4\pi)^2 e_a^2 n_a}{k^2}\int d\boldsymbol{p}\; 2\pi\,\delta(\omega-\boldsymbol{k}\cdot\boldsymbol{v}) f_a \tag{39.32}$$

$$(\delta \boldsymbol{E}^{\perp}\cdot\delta \boldsymbol{E}^{\perp})^{\mathrm{source}}_{\omega,\boldsymbol{k}} = \sum_a \frac{(4\pi)^2 e_a^2 n_a}{k^2}\int d\boldsymbol{p}\; 2\pi\,\delta(\omega-\boldsymbol{k}\cdot\boldsymbol{v})\,\frac{|\boldsymbol{k}\times\boldsymbol{v}|^2}{\omega^2} f_a \tag{39.33}$$

which follow from (39.29), (39.27), (39.20), and (34.11).

In an isotropic plasma, the transverse and longitudinal components cannot be correlated with each other, therefore the tensor $(\delta E_i\, \delta E_j)_{\omega,\boldsymbol{k}}$ can be represented as follows:

$$(\delta E_i\, \delta E_j)_{\omega,\boldsymbol{k}} = (\delta E_i^{\parallel}\, \delta E_j^{\parallel})_{\omega,\boldsymbol{k}} + (\delta E_i^{\perp}\, \delta E_j^{\perp})_{\omega,\boldsymbol{k}} \tag{39.34}$$

where

$$(\delta E_i^{\parallel}\, \delta E_j^{\parallel})_{\omega,\boldsymbol{k}} = \frac{k_i k_j}{k^2}\,(\delta \boldsymbol{E}^{\parallel}\cdot\delta \boldsymbol{E}^{\parallel})_{\omega,\boldsymbol{k}}$$

$$(\delta E_i^{\perp}\, \delta E_j^{\perp})_{\omega,\boldsymbol{k}} = \tfrac{1}{2}\left(\delta_{ij} - \frac{k_i k_j}{k^2}\right)(\delta \boldsymbol{E}^{\perp}\cdot\delta \boldsymbol{E}^{\perp})_{\omega,\boldsymbol{k}}\ . \tag{39.35}$$

Thus, in the isotropic case, the two spectral densities (39.30) and (39.31) fully define the tensor $(\delta E_i\ \delta E_j)_{\omega,\boldsymbol{k}}$. Note that (39.30) coincides with (35.6).

We now derive an expression for the spectral density $(\delta N_a \delta \boldsymbol{E})_{\omega,\boldsymbol{k},\boldsymbol{p}}$ which defines the collision integral. We therefore multiply (39.3) by $\delta E_i(\omega,\boldsymbol{k})$ and average. We note that, through Eqs (39.28),(39.27), (39.20), the function $\delta \boldsymbol{E}(\omega,\boldsymbol{k})$ can be expressed in terms of $\delta N_a^{\mathrm{source}}$:

$$\delta \boldsymbol{E}(\omega,\boldsymbol{k}) = -\frac{4\pi i}{\omega}\sum_a e_a \int d\boldsymbol{p}\left\{\frac{\boldsymbol{k}(\boldsymbol{k}\cdot\boldsymbol{v})}{k^2\varepsilon^{\parallel}(\omega,\boldsymbol{k})} + \frac{\omega^2}{k^2}\frac{(\boldsymbol{k}\times\boldsymbol{v})\times\boldsymbol{k}}{\omega^2\varepsilon^{\perp}(\omega,\boldsymbol{k}) - c^2k^2}\right\}\delta N_a^{\mathrm{source}}(\omega,\boldsymbol{k},\boldsymbol{p}) \tag{39.36}$$

Using this equation, we express the spectral density $(\delta N_a\ \delta \boldsymbol{E})_{\omega,\boldsymbol{k},\boldsymbol{p}}$ in terms of $(\delta N_a\ \delta N_b)^{\mathrm{source}}_{\omega,\boldsymbol{k},\boldsymbol{p},\boldsymbol{p}'}$ defined by (34.11) :

$$\begin{aligned}(\delta N_a \delta E_i)_{\omega,\boldsymbol{k},\boldsymbol{p}} &= -\frac{i e_a n_a}{\omega-\boldsymbol{k}\cdot\boldsymbol{v}+i\Delta}(\delta E_j\,\delta E_i)_{\omega,\boldsymbol{k}}\frac{\partial f_a}{\partial p_j}\\ &+ i\,\frac{4\pi e_a n_a}{k^2}\,2\pi\delta(\omega-\boldsymbol{k}\cdot\boldsymbol{v})\left\{\frac{\boldsymbol{k}}{\varepsilon^{\parallel}(\omega,\boldsymbol{k})} + \frac{\omega\,(\boldsymbol{k}\times\boldsymbol{v})\times\boldsymbol{k}}{\omega^2\varepsilon^{\perp}(\omega,\boldsymbol{k}) - c^2k^2}\right\}_i f_a\end{aligned} \tag{39.37}$$

from which results

$$\begin{aligned}\mathrm{Re}\,(\delta N_a\,\delta E_i)_{\omega,\boldsymbol{k}} &= -\pi e_a n_a\,\delta(\omega-\boldsymbol{k}\cdot\boldsymbol{v})(\delta E_j\,\delta E_i)_{\omega,\boldsymbol{k}}\frac{\partial f_a}{\partial p_j}\\ &- \frac{4\pi e_a n_a}{k^2}\,2\pi\delta(\omega-\boldsymbol{k}\cdot\boldsymbol{v})\left\{\frac{\boldsymbol{k}\,\mathrm{Im}\,\varepsilon^{\parallel}(\omega,\boldsymbol{k})}{|\varepsilon^{\parallel}(\omega,\boldsymbol{k})|^2} + \right.\\ &\left. + \frac{\omega^3[(\boldsymbol{k}\times\boldsymbol{v})\times\boldsymbol{k}]\,\mathrm{Im}\,\varepsilon^{\perp}(\omega,\boldsymbol{k})}{|\omega^2\varepsilon^{\perp}(\omega,\boldsymbol{k}) - c^2k^2|^2}\right\}_i f_a\ .\end{aligned} \tag{39.38}$$

In a similar way we obtain an expression for the more general spectral density:

$$\begin{aligned}(\delta N_a\,\delta N_b)_{\omega,\boldsymbol{k},\boldsymbol{p},\boldsymbol{p}'} &= \delta_{ab}\,\delta(\boldsymbol{p}-\boldsymbol{p}')\,2\pi\delta(\omega-\boldsymbol{k}\cdot\boldsymbol{v})\,n_a f_a\\ &+ \frac{e_a e_b n_a n_b(\delta E_i\,\delta E_j)_{\omega,\boldsymbol{k}}}{(\omega-\boldsymbol{k}\cdot\boldsymbol{v}+i\Delta)(\omega-\boldsymbol{k}\cdot\boldsymbol{v}'-i\Delta)}\frac{\partial f_a}{\partial p_i}\frac{\partial f_b}{\partial p'_j}\\ &- \frac{4\pi e_a e_b n_a n_b\,2\pi\delta(\omega-\boldsymbol{k}\cdot\boldsymbol{v}')}{k^2(\omega-\boldsymbol{k}\cdot\boldsymbol{v}+i\Delta)}\left\{\frac{k_i}{\varepsilon^{\parallel}(\omega,\boldsymbol{k})} + \frac{\omega[(\boldsymbol{k}\times\boldsymbol{v}')\times\boldsymbol{k}]_i}{\omega^2\varepsilon^{\perp}(\omega,\boldsymbol{k}) - c^2k^2}\right\}\frac{\partial f_a}{\partial p_i} f_b\\ &- \frac{4\pi e_a e_b n_a n_b\,2\pi\delta(\omega-\boldsymbol{k}\cdot\boldsymbol{v})}{k^2(\omega-\boldsymbol{k}\cdot\boldsymbol{v}'-i\Delta)}\left\{\frac{k_i}{\varepsilon^{\parallel *}(\omega,\boldsymbol{k})} + \frac{\omega[(\boldsymbol{k}\times\boldsymbol{v})\times\boldsymbol{k}]_i}{\omega^2\varepsilon^{\perp *}(\omega,\boldsymbol{k}) - c^2k^2}\right\}\frac{\partial f_b}{\partial p'_i} f_a\end{aligned} \tag{39.39}$$

We see that the spectral densities $(\delta N_a \delta E)_{\omega,\boldsymbol{k},\boldsymbol{p}}$ and $(\delta N_a \delta N_b)_{\omega,\boldsymbol{k},\boldsymbol{p},\boldsymbol{p}'}$ are expressed in terms of the spectral density of the electric field fluctuations $(\delta E_i\, \delta E_j)_{\omega,\boldsymbol{k}}$ and of the dielectric constants $\varepsilon^{\parallel}, \varepsilon^{\perp}$.

Integrating these expressions over ω we can find the spatial spectral densities; for instance

$$(\delta N_a\, \delta N_b)_{\boldsymbol{k},\boldsymbol{p},\boldsymbol{p}'} = (2\pi)^{-1} \int d\omega\, (\delta N_a\, \delta N_b)_{\omega,\boldsymbol{k},\boldsymbol{p},\boldsymbol{p}'} .$$

We now consider several concrete examples, for the case of an equilibrium state. We substitute into (39.30), (39.31) the Maxwell distribution and use the expressions :

$$\begin{aligned}
\operatorname{Im} \varepsilon^{\parallel} &= -\sum_a \frac{4\pi^2 e_a^2 n_a}{k^2} \int d\boldsymbol{p}\ \delta(\omega - \boldsymbol{k}\cdot\boldsymbol{v})\ \boldsymbol{k}\cdot\frac{\partial f_a}{\partial \boldsymbol{p}} \\
\operatorname{Im} \varepsilon^{\perp} &= -\sum_a \frac{2\pi^2 e_a^2 n_a}{\omega k^2} \int d\boldsymbol{p}\ \delta(\omega - \boldsymbol{k}\cdot\boldsymbol{v})[(\boldsymbol{k}\times\boldsymbol{v})\times\boldsymbol{k}]\cdot\frac{\partial f_a}{\partial \boldsymbol{p}}
\end{aligned} \tag{39.40}$$

which follow from (39.24), (39.25). We thus obtain :

$$(\delta \boldsymbol{E}^{\parallel}\cdot\delta \boldsymbol{E}^{\parallel})_{\omega,\boldsymbol{k}} = \frac{8\pi}{\omega}\,\frac{\operatorname{Im}\varepsilon^{\parallel}(\omega,\boldsymbol{k})}{|\varepsilon^{\parallel}(\omega,\boldsymbol{k})|^2}\ k_{\mathrm{B}}T \quad , \tag{39.41}$$

$$(\delta \boldsymbol{E}^{\perp}\cdot\delta \boldsymbol{E}^{\perp})_{\omega,\boldsymbol{k}} = \frac{16\pi\omega^3\ \operatorname{Im}\varepsilon^{\perp}(\omega,\boldsymbol{k})}{|\omega^2\varepsilon^{\perp}(\omega,\boldsymbol{k}) - c^2k^2|^2}\ k_{\mathrm{B}}T \ . \tag{39.42}$$

Clearly, (39.41) is identical with (36.4). For small k one obtains (36.32) from the latter equation. Similarly, one finds from (39.42), for small k, (in the region of transparency) :

$$(\delta \boldsymbol{E}^{\perp}\cdot\delta \boldsymbol{E}^{\perp})_{\omega,\boldsymbol{k}} = 16\pi^2(\operatorname{sign}\omega)\ \delta(\omega^2\operatorname{Re}\varepsilon^{\perp} - c^2k^2)\ k_{\mathrm{B}}T \tag{39.43}$$

or

$$(\delta \boldsymbol{E}^{\perp}\cdot\delta \boldsymbol{E}^{\perp})_{\omega,\boldsymbol{k}} = 8\pi^2[\delta(\omega-\omega_k) + \delta(\omega+\omega_k)]\ k_{\mathrm{B}}T \tag{39.44}$$

where $\omega_k = (\omega_L^2 + c^2k^2)^{\frac{1}{2}}$.

For the spatial spectral density, we find from (39.44) :

$$(\delta \boldsymbol{E}^{\perp}\cdot\delta \boldsymbol{E}^{\perp})_{\boldsymbol{k}} = 8\pi\ k_{\mathrm{B}}T \ . \tag{39.45}$$

This result is analogous to Eq. (36.37) for the longitudinal field, which is valid for $k \ll 1/r_{\mathrm{D}}$. We now show that, contrary to the latter, the result (39.45) is valid for arbitrary k. We therefore integrate (39.42) over ω and write the result in the following form :

$$(\delta \boldsymbol{E}^{\perp}\cdot\delta \boldsymbol{E}^{\perp})_{\boldsymbol{k}} = -8\pi\ k_{\mathrm{B}}T\ \frac{1}{2\pi i}\int d\omega\ \omega\left[\frac{1}{\omega^2\varepsilon^{\perp} - c^2k^2} - c.c.\right] . \tag{39.46}$$

The first integrand has singularities only in the lower half-plane, and the second in the upper one. We therefore close the contour of integration in the lower half-

plane for the first, and in the upper one for the second term. Noting that $\omega^2 \varepsilon^{\perp} - c^2 k^2 \to \infty$ for $\omega \to \infty$ and using the result :

$$-\frac{1}{2\pi i} \int d\omega \; \frac{1}{\omega \pm i\Delta} = \pm \frac{1}{2}$$

we find from (39.46) the result (39.45), without any approximation.

This result is due to the following circumstances. The integral (39.46) is defined by the values of the function $\varepsilon^{\perp}(\omega, \boldsymbol{k})$ for $\omega \to \infty$, but for fixed k. Thus, $\omega/k > c$, i.e., the phase velocity exceeds the velocity of light. From (39.40) follows that $\operatorname{Im} \varepsilon^{\perp}(\omega, \boldsymbol{k}) \to 0$ for $\Delta \to 0$, if $\omega/k > c$. Indeed, the condition $\omega = \boldsymbol{k} \cdot \boldsymbol{v}$ required by the δ-function cannot be satisfied, as there are no particles with speeds superior to c. Thus, in equilibrium, (39.46) yields a contribution only in the region of transparency.

In the equilibrium state, the fluctuations of the transverse field do not affect the correlation function. Let us, indeed, substitute the Maxwell distribution into (39.39) and integrate over ω. All the terms containing the fluctuations of the transverse field vanish and we get for the correlations :

$$g_{ab}(\boldsymbol{k}, \boldsymbol{p}, \boldsymbol{p}') = \left\{ (\delta N_a \, \delta N_b)_{\boldsymbol{k}, \boldsymbol{p}, \boldsymbol{p}'} - n_a \delta_{ab} \, \delta(\boldsymbol{p} - \boldsymbol{p}') f_a \right\} \; n_a n_b$$

the expression

$$g_{ab}(\boldsymbol{k}, \boldsymbol{p}, \boldsymbol{p}') = -\frac{e_a e_b}{\sum\limits_c e_c^2 n_c (1 + r_{\mathrm{D}}^2 k^2)} f_a(\boldsymbol{p}) \, f_b(\boldsymbol{p}') \,. \tag{39.47}$$

This expression remains valid even for relativistic plasmas, if we take for f_a, f_b the relativistic equilibrium distribution (sect. 40).

Equation (39.47) coincides with (36.22) for the Coulomb plasma.

In the presence of an external field, the functions f_a are no longer isotropic. In the anisotropic case, the tensor ε_{ij} depends not only on $\boldsymbol{k}$ but also on other vectors, hence it is no longer defined by the two scalars $\varepsilon^{\parallel}$, $\varepsilon^{\perp}$ alone. Let us note the changes which occur in (39.3) – (39.7) as a result of the anisotropy.

In (39.3) we may no longer suppress the term $(\boldsymbol{v} \times \delta \boldsymbol{B}) \cdot (\partial f_a / \partial \boldsymbol{p})$, therefore the equation for $\delta N_a(\omega, \boldsymbol{k}, \boldsymbol{p})$ becomes

$$\begin{aligned} \delta N_a(\omega, \boldsymbol{k}, \boldsymbol{p}) &= \delta N_a^{\text{source}}(\omega, \boldsymbol{k}, \boldsymbol{p}) \\ &\quad - \frac{i e_a n_a}{\omega - \boldsymbol{k} \cdot \boldsymbol{v} + i\Delta} \left\{ \delta \boldsymbol{E}(\omega, \boldsymbol{k}) + \frac{1}{c} [\boldsymbol{v} \times \delta \boldsymbol{B}(\omega, \boldsymbol{k})] \right\} \cdot \frac{\partial f_a}{\partial \boldsymbol{p}} \,. \end{aligned} \tag{39.48}$$

Therefore the dielectric tensor is defined now by

$$\varepsilon_{ij} = \delta_{ij} + \sum_{\alpha} \frac{4\pi e_a^2 n_a}{\omega} \int d\boldsymbol{p} \; \frac{1}{\omega - \boldsymbol{k} \cdot \boldsymbol{v} + i\Delta} \left\{ v_i \left(1 - \frac{\boldsymbol{k} \cdot \boldsymbol{v}}{\omega} \right) \frac{\partial f_a}{\partial p_j} + \frac{v_i v_j}{\omega} \boldsymbol{k} \cdot \frac{\partial f_a}{\partial \boldsymbol{p}} \right\} . \tag{39.49}$$

In the expression of the tensor $(\delta E_i \delta E_j)_{\omega, \boldsymbol{k}}$ appears the tensor A_{ij}^{-1}, the inverse of

$$A_{ij} = \frac{\omega^2}{c^2}\,\varepsilon_{ij} - (k^2\,\delta_{ij} - k_i\,k_j)\ . \tag{39.50}$$

In the isotropic case:

$$A^{-1}_{ij} = \frac{c^2}{\omega^2 k^2}\left\{\frac{k_i\,k_j}{\varepsilon^{\parallel}} + \frac{\delta_{ij}\,k^2 - k_i\,k_j}{\varepsilon^{\perp} - c^2\,k^2/\omega^2}\right\}\ . \tag{39.51}$$

The calculation of the spectral densities proceeds as before.

40. KINETIC EQUATION FOR A RELATIVISTIC PLASMA

We consider two forms of the collision integral of a relativistic plasma.

We substitute (39.38) into the second expression (33.5) and write the result in the form (37.4) :

$$I_a = \frac{\partial}{\partial p_i}\,D^a_{ij}\,\frac{\partial f_a}{\partial p_j} + \frac{\partial}{\partial p_i}\left(A^a_i\,f_a\right). \tag{40.1}$$

The coefficients D^a_{ij}, A^a_i are then defined as follows :

$$D^a_{ij} = \frac{e_a^2}{16\pi^3}\int d\omega\,d\boldsymbol{k}\;\delta(\omega - \boldsymbol{k}\cdot\boldsymbol{v})\left\{\left(\delta E^{\parallel}_i\,\delta E^{\parallel}_j\right)_{\omega,\boldsymbol{k}} + \left(\delta E^{\perp}_i\,\delta E^{\perp}_j\right)_{\omega,\boldsymbol{k}}\right\} \tag{40.2}$$

$$A^a_i = \frac{e_a^2}{2\pi^2}\int d\omega\,d\boldsymbol{k}\;\delta(\omega - \boldsymbol{k}\cdot\boldsymbol{v})\left\{\frac{k_i\,\mathrm{Im}\,\varepsilon^{\parallel}(\omega,\boldsymbol{k})}{k^2|\varepsilon^{\parallel}(\omega,\boldsymbol{k})|^2} + \frac{\omega^3[(\boldsymbol{k}\times\boldsymbol{v})\times\boldsymbol{k}]_i\,\mathrm{Im}\,\varepsilon^{\perp}(\omega,\boldsymbol{k})}{k^2|\omega^2\,\varepsilon^{\perp}(\omega,\boldsymbol{k}) - c^2\,k^2|^2}\right\}. \tag{40.3}$$

The functions $(\delta E^{\parallel}_i\,\delta E^{\parallel}_j)_{\omega,\boldsymbol{k}}$, $(\delta E^{\perp}\,\delta E^{\perp}_j)_{\omega,\boldsymbol{k}}$, $\varepsilon^{\parallel}$, $\varepsilon^{\perp}$ are defined by Eqs (39.30), (39.31), (39.35), (39.24), (39.25). In the non-relativistic limit, the contribution of the transverse field fluctuations can be neglected and (40.2), (40.3) reduce to (37.5), (37.6) .

The second form of the collision integral corresponds to the non-relativistic form (37.1). We eliminate the spectral densities and the imaginary parts of the dielectric constants by using (39.30), (39.31), (39.35), (39.40), and integrate over ω. We then write :

$$I_a = \sum_b n_b\,\frac{\partial}{\partial p_i}\int d\boldsymbol{p}'\;Q^{ab}_{ij}\left\{\frac{\partial f_a}{\partial p_j}\,f_b - \frac{\partial f_b}{\partial p'_j}\,f_a\right\} \tag{40.4}$$

where

$$Q^{ab}_{ij} = 2\,e_a^2\,e_b^2\int d\boldsymbol{k}\;\frac{k_i\,k_j}{k^4}\,\delta(\boldsymbol{k}\cdot\boldsymbol{v} - \boldsymbol{k}\cdot\boldsymbol{v}')$$
$$\cdot\left[\frac{1}{|\varepsilon^{\parallel}(\omega,\boldsymbol{k})|^2} + \frac{[(\boldsymbol{k}\times\boldsymbol{v})\cdot(\boldsymbol{k}\times\boldsymbol{v}')]^2}{|(\boldsymbol{k}\cdot\boldsymbol{v})^2\,\varepsilon^{\perp}(\boldsymbol{k}\cdot\boldsymbol{v},\boldsymbol{k}) - c^2\,k^2|^2}\right]\ . \tag{40.5}$$

We used here the identity

$$[(\boldsymbol{k}\times\boldsymbol{v})\times\boldsymbol{k}]\cdot\frac{\partial f_a}{\partial\boldsymbol{p}}\,(\boldsymbol{k}\cdot\boldsymbol{v}') = [(\boldsymbol{k}\times\boldsymbol{v})\cdot(\boldsymbol{k}\times\boldsymbol{v}')]\,\boldsymbol{k}\cdot\frac{\partial f_a}{\partial\boldsymbol{p}}\ . \tag{40.6}$$

In the non-relativistic limit, the second term in (40.5) is negligible, and (40.4), (40.5) reduce to (37.1).

Equation (40.4) thus represents the relativistic generalization of the Balescu-Lenard equation; it was derived by Silin [31].

Equation (40.4) is quite complicated. It is therefore useful to obtain simpler expressions of the collision integral, in which the plasma polarization is treated more crudely. In section 38 it was shown that for states not too far from equilibrium, one may use, instead of (37.1), the simpler Landau equation. An analogous simplification can be obtained in the relativistic case too.

We first note that the stationary solution of the kinetic equation (40.4) is the relativistic Jüttner distribution

$$f_a(\boldsymbol{p}) = A_a \exp\left\{-\frac{c(p^2+m^2c^2)^{\frac{1}{2}}}{k_\mathrm{B}T}\right\}. \tag{40.7}$$

The constant A_a is obtained from the normalization condition. This statement is easily checked, by noting that for this function

$$\frac{\partial}{\partial \boldsymbol{p}} = \frac{\partial \varepsilon}{\partial \boldsymbol{p}}\frac{\partial}{\partial \varepsilon} = \boldsymbol{p}\,\frac{c^2}{\varepsilon}\frac{\partial}{\partial \varepsilon} = \boldsymbol{v}\,\frac{\partial}{\partial \varepsilon}$$

because

$$\boldsymbol{p} = \frac{\varepsilon}{c^2}\boldsymbol{v}\ .$$

Hence (40.4) becomes

$$I_a = -\frac{1}{k_\mathrm{B}T}\sum_b n_b \frac{\partial}{\partial p_i}\int d\boldsymbol{p}'\, Q_{ij}^{ab}(\boldsymbol{v}-\boldsymbol{v}')_j\, f_a(\boldsymbol{p})\, f_b(\boldsymbol{p}')\,.$$

This expression vanishes because of the δ-function in Q_{ij}^{ab}.

We now come back to the problem of the simplification of the collision integral. It was shown in section 39 that the transverse field fluctuations do not affect the form of the correlation function $g_{ab}(\boldsymbol{k},\boldsymbol{p},\boldsymbol{p}')$. Indeed, (39.47) differs from (36.22) only in the fact that the distribution functions appearing in the former are relativistic. Thus, the expression of $g_{ab}(\boldsymbol{r},\boldsymbol{p},\boldsymbol{p}')$ coincides with (36.24). The radius of correlation is therefore of order r_D in the relativistic case as well. This implies that the spatial spectrum is limited, in equilibrium, to small values of k, with $k_{\mathrm{min}} \sim r_\mathrm{D}^{-1}$. This statement remains valid also for states close to equilibrium. We may therefore set approximately in (40.4), (40.5) : $\varepsilon^{\parallel} = 1$, $\varepsilon^{\perp} = 1$, and integrate only over values of k in the range $k > r_\mathrm{D}^{-1}$. As a result we obtain :

$$Q_{ij}^{ab} = 2e_a^2e_b^2\int d\boldsymbol{k}\ \delta(\boldsymbol{k}\cdot\boldsymbol{v}-\boldsymbol{k}\cdot\boldsymbol{v}')\,\frac{k_ik_j}{k^4}\left\{1+\frac{[(\boldsymbol{k}\times\boldsymbol{v})\cdot(\boldsymbol{k}\times\boldsymbol{v}')]^2}{|(\boldsymbol{k}\cdot\boldsymbol{v})^2-c^2k^2|^2}\right\}$$

$$(k > r_\mathrm{D}^{-1})\ . \tag{40.8}$$

We must also cut off the integration for large values of k, as in the Landau case. The value $k_{\mathbf{max}}$ can be defined by a condition analogous to (38.2).

For an isotropic plasma, (40.8) can be written as

$$Q_{ij}^{ab} = 2\, e_a^2 e_b^2 \int d\boldsymbol{k}\; \delta(\boldsymbol{k}\cdot\boldsymbol{v} - \boldsymbol{k}\cdot\boldsymbol{v}')\, \frac{k_i k_j}{k^4} \left\{1 + \frac{(\boldsymbol{k}\times\boldsymbol{v})\cdot(\boldsymbol{k}\times\boldsymbol{v}')}{(\boldsymbol{k}\cdot\boldsymbol{v})^2 - c^2 k^2}\right\}^2 . \tag{40.9}$$

This expression differs from (40.8) only by a term which vanishes by integration over $\boldsymbol{p}'$. After a few simplifications, this equation can be written in the form [8,9] :

$$Q_{ij}^{ab} = 2\, e_a^2 e_b^2 \left[c^2 - (\boldsymbol{v}\cdot\boldsymbol{v}')\right]^2 \int d\boldsymbol{k}\; \delta(\boldsymbol{k}\cdot\boldsymbol{v} - \boldsymbol{k}\cdot\boldsymbol{v}')\, \frac{k_i k_j}{[c^2 k^2 - (\boldsymbol{k}\cdot\boldsymbol{v})^2]^2} \tag{40.10}$$

In the limit $c\to\infty$, (40.10) reduces to (38.4).

The kinetic equation with the collision integral (40.4), (40.8) (or (40.10)) is the relativistic generalization of the Landau equation for the 'ordinary' distribution function in the six-dimensional phase space, $f_a(\boldsymbol{r}, \boldsymbol{p}, t)$. One may also derive an equation for the distribution function

$$n_a\, F_a(Q, P) \equiv \langle N_a(Q, P)\rangle \tag{40.11}$$

for the four-vectors $Q = (\boldsymbol{r}, ict)$, $P = (\boldsymbol{p}, i\mathcal{E}/c)$. The microscopic function $N_a(Q,P)$ is defined by Eqs (24.16) – (24.18). The kinetic equation for F_a is [32 , 9 , 8] :

$$U_i \frac{\partial F_a}{\partial Q_i} = \sum_b n_b \int d\boldsymbol{p}'\, d\mathcal{E}'\; \frac{\partial}{\partial P_i}\, \mathbb{C}_{ij}^{ab} \left\{\frac{\partial F_a}{\partial P_j} F_b - \frac{\partial F_b}{\partial P_j'} F_a\right\} \tag{40.12}$$

with

$$\mathbb{C}_{ij}^{ab} = 2\, e_a^2 e_b^2 (U_n U_n')^2 \int d\boldsymbol{k}\, d\omega\; \delta(k_m U_m)\; \delta(k_m U_m')\, \frac{k_i k_j}{(c^2 k^2 - \omega^2)} \tag{40.13}$$

In these equations $U = (\gamma\boldsymbol{v}, i\gamma c)$ is the four-velocity, $P_i = m_a U_i$, $k = (\boldsymbol{k}, i\omega/c)$ is the wave four-vector.

The integrations over ω and $\boldsymbol{k}$ can be performed explicitly in these equations. We have to use the symmetry properties of the tensor $\mathbb{C}_{ij}^{ab}$ with respect to U_i, U_i' and the relations

$$U_i \mathbb{C}_{ij}^{ab} = U_i' \;\mathbb{C}_{ij}^{ab} = 0 \;.$$

We then obtain

$$\begin{aligned}\mathbb{C}_{ij}^{ab} = \frac{2\, e_a^2 e_b^2}{c^5} \left(\ln \frac{r_{\mathbf{D}}}{r_{\mathbf{min}}}\right) \Big\{\Big[\big(U_l U_l'\big)^2 / c^4 - 1\Big]\, \delta_{ij} \\ - (U_i U_j + U_i' U_j')/c^2 - (U_l U_l')(U_i U_j' + U_i' U_j)/c^4\Big\} \\ \cdot \Big\{\big(U_l U_l'\big)^2 / c^4 - 1\Big\}^{-\frac{3}{2}}\end{aligned} \tag{40.14}$$

where $r_{\mathbf{min}} = k_{\mathbf{max}}^{-1}$. The kinetic equation (40.12), with the kernel (40.14) was first derived by Belyaev and Budker [32 , 9 , 8] .

In equilibrium, the kinetic equation (40.12) has the solution

$$F_a(P) = C\ \delta(P_i^2 + m_a^2 c^2)\ \exp\left(\frac{P_i U_i}{k_B T}\right) \tag{40.15}$$

which is the four-dimensional equilibrium distribution function.

In order to go over from (40.12) to the kinetic equation for the function f_a, we must use the relation:[†]

$$F_a(Q,P) = f_a(\boldsymbol{r}, \boldsymbol{p}, t)\ \delta[\mathcal{E} - c(p^2 + m_a^2 c^2)^{\frac{1}{2}}]\,(m_a c^2/\mathcal{E}) \tag{40.16}$$

† Different treatments of the kinetic theory of relativistic plasmas can be found in references [63, 64, 65, 66]. (Trans.)

CHAPTER 8

Effect of an External Field on the Kinetic Properties of Plasmas

41. EQUATIONS FOR THE FUNCTIONS f_a, δN_a IN THE PRESENCE OF AN EXTERNAL FIELD

We investigate the influence of an external electric field on the kinetic properties of a spatially homogeneous plasma. In that case Eqs (26.5) and (27.7) for the functions f_a, δN_a take the following form:

$$\frac{\partial f_a}{\partial t} + e_a \boldsymbol{E}(t) \cdot \frac{\partial f_a}{\partial \boldsymbol{p}} = -\frac{e_a}{n_a} \frac{\partial}{\partial \boldsymbol{p}} \cdot \frac{1}{(2\pi)^3} \int d\boldsymbol{k}\, \mathrm{Re}(\delta N_a \delta \boldsymbol{E})_{\boldsymbol{k},\boldsymbol{p},t} \equiv I_a(\boldsymbol{p}, t) \tag{41.1}$$

$$\left(\frac{\partial}{\partial t} + \Delta + \boldsymbol{v} \cdot \frac{\partial}{\partial \boldsymbol{r}} + e_a \boldsymbol{E}(t) \cdot \frac{\partial}{\partial \boldsymbol{p}}\right)\left(\delta N_a - \delta N_a^{\text{source}}\right) = -e_a \delta \boldsymbol{E} \cdot \frac{\partial n_a f_a}{\partial \boldsymbol{p}}\,,$$
$$\nabla \times \delta \boldsymbol{E} = 0\,, \quad \nabla \cdot \delta \boldsymbol{E} = \sum_a 4\pi e_a \int d\boldsymbol{p}\, \delta N_a\,. \tag{41.2}$$

The correlations of the fluctuations $\delta N_a^{\text{source}}$ are defined by

$$\left(\frac{\partial}{\partial t} + \Delta + \boldsymbol{v} \cdot \frac{\partial}{\partial \boldsymbol{r}} + e_a \boldsymbol{E}(t) \cdot \frac{\partial}{\partial \boldsymbol{p}}\right) \langle \delta N_a \delta N_b \rangle^{\text{source}}_{x,t,x',t'} = 0\,, \tag{41.3}$$

which must be solved with the initial condition

$$\langle \delta N_a \delta N_b \rangle^{\text{source}}_{x,t,x',t'} \Big|_{t=t'} = n_a \delta_{ab} \delta(x' - x) f_a(x', t')\,. \tag{41.4}$$

As a result of the external field, the distribution function $f_a(\boldsymbol{p}, t)$ may not necessarily be slowly varying. This fact seriously complicates the calculation of the spectral density of fluctuations of δN_a, $\delta \boldsymbol{E}$. In the case of a spatially homogeneous plasma, this calculation can be significantly simplified if, instead of the distribution function $f_a(\boldsymbol{p}, t)$ we introduce a new function:

$$f_a\left(\boldsymbol{P}_a + e_a \int_{-\infty}^{t} dt'\, \boldsymbol{E}(t'), t\right) \equiv F_a(\boldsymbol{P}_a, t)\,. \tag{41.5}$$

From (41.1) we obtain the following equation for the function F_a:

$$\frac{\partial F_a}{\partial t} = I_a\left(P_a + e_a \int_{-\infty}^{t} dt' \, E(t'), t\right) \equiv I_a(P_a, t) \tag{41.6}$$

The collision integral $I_a(P_a, t)$ is related to the spectral density $(\delta N_a \delta E)_{k,p,t}$ in the following way

$$I_a(P_a, t) = -\frac{e_a}{n_a(2\pi)^3} \frac{\partial}{\partial P_a} \cdot \int dk \, \mathrm{Re}\left(\delta N_a \delta E\right)_{k, P_a + e_a \int_{-\infty}^{t} dt' \, E(t')} \,. \tag{41.7}$$

From Eqs (41.6) and (41.7) one can see that, to first order in the plasma parameter, [the basic equations (41.1) – (41.4) actually correspond to that approximation] the rate of change of the distribution function is of the order $\mu \omega_L F_a$. As a result, the fast oscillating contributions to the function F_a are small if $\mu \ll 1$. The main role will be played by the slowly varying part of the distribution function. If the external field is periodic, with frequency ω_0, then, for $\omega_0 \gtrsim \omega_L$, the slowly varying part of the distribution function is obtained from $F_a(P_a, t)$ by averaging over a period $2\pi/\omega_0$.

Below, in the absence of any explicit statement, we shall denote by F_a, I_a the slowly varying parts of those functions.

From Eqs (41.2) – (41.4) follows that the calculation of the spectral density defining the collision integral (41.7) requires the knowledge of the spectral density of the fluctuations δE^{source}, $\delta N_a^{\text{source}}$.

42 SPECTRAL DENSITY OF THE SOURCE FLUCTUATIONS IN THE PRESENCE OF A HIGH-FREQUENCY ELECTRIC FIELD

The spectral density of the fluctuations $\delta N_a^{\text{source}}$ is defined by Eq. (41.3) with the initial condition (41.4). In the presence of a high-frequency field, the fluctuation process is non-stationary, even for an ideal plasma. This fact makes the solution of the problem difficult.

We start from the equation of motion

$$\frac{dp}{dt} = e_a E(t) \tag{42.1}$$

The solution to the initial-value problem defined above is:

$$\begin{aligned}\langle \delta N_a \, \delta N_b \rangle^{\text{source}}_{x,t,x',t-\tau} &= n_a \delta_{ab} \delta\left(p - p' - e_a \int_{t-\tau}^{t} dt' \, E(t')\right) \\ &\quad \delta\left(r - r' - v\tau - \frac{e_a}{m_a} \int_{t-\tau}^{t} dt' \, (t-\tau-t') E(t')\right) \\ &\quad f_a\left(p - e_a \int_{t-\tau}^{t} dt' \, E(t'), \, t-\tau\right)\end{aligned} \tag{42.2}$$

Note that, from (41.5)

$$f_a\left(\boldsymbol{p}-e_a \int\limits_{t-\tau}^{t} dt'\, \boldsymbol{E}(t'),\, t-\tau\right)$$
$$= f_a\left(\boldsymbol{P}_a + e_a \int\limits_{-\infty}^{t} dt'\, \boldsymbol{E}(t') - e_a \int\limits_{t-\tau}^{t} dt'\, \boldsymbol{E}(t'),\, t-\tau\right) = F_a(\boldsymbol{P}_a\,,\, t-\tau)\,. \quad (42.3)$$

Thus the function (42.2) can be expressed in terms of the slowly varying function F_a. For a nonideal plasma the time-delay in the function F_a can be neglected. In this approximation we find from (42.2), (42.3) the desired spectral density:

$$\left(\delta N_a\, \delta N_b\right)^{\text{source}}_{\omega,\boldsymbol{k},\boldsymbol{p},\boldsymbol{p}',t}$$
$$= 2\, n_a\, \delta_{ab}\, \mathrm{Re} \int\limits_0^\infty d\tau\, \delta\left[\boldsymbol{p}-\boldsymbol{p}'-e_a \int\limits_{t-\tau}^{t} dt'\, \boldsymbol{E}(t')\right]$$
$$\cdot \exp\left\{-\Delta\tau + i(\omega-\boldsymbol{k}\cdot\boldsymbol{v})\tau - i\,\frac{e_a}{m_a}\int\limits_{t-\tau}^{t} dt'\,(t-\tau-t')\,\boldsymbol{k}\cdot\boldsymbol{E}(t')\right\}$$
$$\cdot F_a\,(\boldsymbol{P}_a\,, t) \quad (42.4)$$

For $\boldsymbol{E}(t) = 0$ this result reduces to (34.11).

From (42.4) follows the expression for the simpler function:

$$\left(\delta N_a\;\delta \boldsymbol{E}\right)^{\text{source}}_{\omega,\boldsymbol{k},\boldsymbol{p},t} = i\,\frac{4\pi\, e_a\, n_a}{k^2}\, 2\boldsymbol{k}$$
$$\cdot\, \mathrm{Re}\int\limits_0^\infty d\tau\, \exp\left\{-\Delta\tau + i(\omega-\boldsymbol{k}\cdot\boldsymbol{v})\,\tau - i\,\frac{e_a}{m_a}\int\limits_{t-\tau}^{t} dt'\,(t-\tau-t')\,\boldsymbol{k}\cdot\boldsymbol{E}(t')\right\}$$
$$\cdot\, F_a(\boldsymbol{P}_a \cdot t) \quad (42.5)$$

and for the spectral density of the fluctuating field:

$$\left(\delta\boldsymbol{E}\cdot\delta\boldsymbol{E}\right)^{\text{source}}_{\omega,\boldsymbol{k},t} = \sum_a \frac{(4\pi)^2\, e_a^2\, n_a}{k^2}$$
$$\cdot\, 2\,\mathrm{Re}\int\limits_0^\infty d\tau \int d\boldsymbol{p}\, \exp\left\{-\Delta\tau + i(\omega-\boldsymbol{k}\cdot\boldsymbol{v})\,\tau - i\,\frac{e_a}{m_a}\int\limits_{t-\tau}^{t} dt'\,(t-\tau-t')\,\boldsymbol{k}\cdot\boldsymbol{E}(t')\right\}$$
$$\cdot\, F_a(\boldsymbol{P}_a\,, t) \quad (42.6)$$

In this expression it is convenient to go over to $\boldsymbol{P}_a$ as an integration variable. Noting that

$$\boldsymbol{k}\cdot\boldsymbol{v}\,\tau + \frac{e_a}{m_a}\int\limits_{t-\tau}^{t} dt'\,(t-\tau-t')\,\boldsymbol{k}\cdot\boldsymbol{E}(t') = \boldsymbol{k}\cdot\boldsymbol{V}_a\,\tau - \frac{e_a}{m_a}\int\limits_{t}^{t-\tau} dt' \int\limits_{-\infty}^{t'} dt''\, \boldsymbol{k}\cdot\boldsymbol{E}(t'')$$

$$\boldsymbol{V}_a = \boldsymbol{P}_a\,/\,m_a \quad (42.7)$$

we can rewrite (42.6) as

$$\left(\delta E \cdot \delta E\right)^{\text{source}}_{\omega, k, t} = \sum_a \frac{(4\pi)^2 e_a^2 n_a}{k^2} \cdot 2\,\mathrm{Re} \int_0^\infty d\tau \int dP_a \exp\left\{-\Delta\tau + i(\omega - k\cdot V_a)\tau + \frac{ie_a}{m_a} k \cdot \int_t^{t-\tau} dt' \int_{-\infty}^{t'} dt''\, E(t'')\right\} \cdot F_a(P_a, t) \tag{42.8}$$

We see that the spectral densities (42.4) – (42.6) depend explicitly on time. This fact expresses the non-stationary character of the processes which are the sources of the fluctuations of the phase densities and of the fields.

For definiteness, we consider below a field of the form

$$E(t) = E \sin\omega_0 t \tag{42.9}$$

Hence

$$\int_t^{t-\tau} dt' \int_{-\infty}^{t'} dt''\, E \sin\omega_0 t'' = \frac{E}{\omega_0^2}\left[\sin\omega_0 t - \sin\omega_0 (t-\tau)\right] \tag{42.10}$$

and the expression (42.8) becomes

$$\left(\delta E \cdot \delta E\right)^{\text{source}}_{\omega, k, t} = \sum_a \frac{(4\pi)^2 e_a^2 n_a}{k^2} \cdot 2\,\mathrm{Re} \int_0^\infty d\tau \int dP \exp\left\{-\Delta\tau + i(\omega - k\cdot v)\tau - i\,a_a\left[\sin\omega_0(t-\tau) - \sin\omega_0 t\right]\right\} \cdot F_a(P, t)\,. \tag{42.11}$$

In Eq. (42.5) we make the change of variables $p = P_a + e_a \int_{-\infty}^{t} dt'\, E(t')$. Then, using Eq. (42.7) we obtain

$$\left(\delta N_a\, \delta E\right)^{\text{source}}_{\omega, k, P_a + e_a \int_{-\infty}^{t} dt'\, E(t')} = i\,\frac{4\pi e_a n_a}{k^2}\, k \cdot 2\,\mathrm{Re} \int_0^\infty d\tau \exp\left\{-\Delta\tau + i(\omega - k\cdot V_a)\tau - i\,\frac{e_a}{m_a} k \int_t^{t-\tau} dt' \int_{-\infty}^{t'} dt''\, E(t'')\right\} \cdot F_a(P_a, t)\,. \tag{42.12}$$

For $E(t)$ of the form (42.9) :

$$\int_{-\infty}^{t} dt'\, E(t') = -\frac{E}{\omega_0} \cos\omega_0 t \tag{42.13}$$

and (42.12) becomes :

$$\left(\delta N_a\,\delta \boldsymbol{E}\right)^{\text{source}}_{\omega,\boldsymbol{k},\boldsymbol{P}_a-(e_a/\omega_0)\,\boldsymbol{E}\cos\omega_0 t} = \frac{4\pi e_a n_a}{k^2}\,i\boldsymbol{k}$$
$$\cdot\, 2\,\mathrm{Re}\int_0^\infty d\tau\,\exp\left\{-\Delta\tau+i\,(\omega-\boldsymbol{k}\cdot\boldsymbol{V}_a)\,\tau-i\,a_a\left[\sin\omega_0(t-\tau)-\sin\omega_0 t\right]\right\}$$
$$\cdot\, F_a(\boldsymbol{P}_a\,,t)\,. \tag{42.14}$$

In Eqs (42.11), (42.14) we introduced the notation

$$a_a = \frac{e_a\,\boldsymbol{k}\cdot\boldsymbol{E}}{m_a\,\omega_0^2}\,. \tag{42.15}$$

We now use Eq. (42.11) for the calculation of the spectral density of the fluctuating field for the case $\omega_0 \gtrsim \omega_L$.

43. SPECTRAL DENSITY OF THE ELECTRIC FIELD FLUCTUATIONS

From the first equation (41.2) we find that, for $\boldsymbol{E}(t) = \boldsymbol{E}\sin\omega_0 t$, the Fourier components of the phase density are defined as follows, to zeroth order in the retardation :

$$\delta N_a\left(\boldsymbol{k}\,,\boldsymbol{P}_a-\frac{e_a}{\omega_0}\cos\omega_0 t\,,t\right) = \delta N_a^{\text{source}}\left(\boldsymbol{k}\,,\boldsymbol{P}_a-\frac{e_a}{\omega_0}\cos\omega_0 t\,,t\right)$$
$$-\,e_a n_a\int_0^\infty d\tau\,\exp\left\{-\Delta\tau-i\boldsymbol{k}\cdot\boldsymbol{V}_a\tau-i\,a_a\left[\sin\omega_0(t-\tau)-\sin\omega_0 t\right]\right\}$$
$$\delta\boldsymbol{E}(\boldsymbol{k}\,,t-\tau)\,\cdot\frac{\partial F_a(\boldsymbol{P}_a,t)}{\partial \boldsymbol{P}_a} \tag{43.1}$$

From the equations (41.2) for the field fluctuations we find :

$$i\boldsymbol{k}\cdot\delta\boldsymbol{E}(\boldsymbol{k}\,,t) = 4\pi\sum_a \delta q_a(\boldsymbol{k}\,,t) = 4\pi\sum_a e_a\int d\boldsymbol{p}\;\delta N_a(\boldsymbol{k}\,,\boldsymbol{p}\,,t) \tag{43.2}$$

Here $\delta q_a(\boldsymbol{k}\,,t)$ denotes the fluctuation of the charge density of component a.

From Eqs (43.1) , (43.2) follows a set of equations for the functions $\delta q_a(\boldsymbol{k}\,,t)$. Using the Bessel-function expansion

$$\exp(-i a_a\sin\omega_0 t) = \sum_{n=-\infty}^{\infty} I_n(a_a)\,\exp(-i n\omega_0 t) \tag{43.3}$$

we write the equations for $\delta q_a(\omega\,,\boldsymbol{k})$ in the form :

$$\delta q_a(\omega,\boldsymbol{k}) = \delta q_a^{\text{source}}(\omega,\boldsymbol{k}) + i\,\frac{4\pi e_a^2 n_a}{k^2}\sum_b\sum_n\sum_m I_n(a_a)\,I_m(a_b)$$
$$\cdot\int_0^\infty d\tau\int d\boldsymbol{P}\,\exp\left\{-\Delta\tau+i(\omega+n\omega_0-\boldsymbol{k}\cdot\boldsymbol{V})\,\tau\right\}\boldsymbol{k}\cdot\frac{\partial F_a(\boldsymbol{P},t)}{\partial \boldsymbol{P}}\delta q_b[\omega+(n-m)\,\omega_0\,,\boldsymbol{k}].$$

These equations can be simplified by introducing new functions instead of $\delta q_a(\boldsymbol{k},t)$, $\delta q_a(\omega,\boldsymbol{k})$:

$$\delta\tilde{q}_a(\boldsymbol{k},t) = e^{-i a_a \sin\omega_0 t}\,\delta q_a(\boldsymbol{k},t) = \sum_n I_n(a_a) e^{-in\omega_0 t}\,\delta q_a(\boldsymbol{k},t)$$

$$\delta\tilde{q}_a(\omega,\boldsymbol{k}) = \sum_n I_n(a_a)\,\delta q_a(\omega - n\omega_0,\boldsymbol{k}). \tag{43.4}$$

The set of equations for $\delta\tilde{q}_a(\omega,\boldsymbol{k})$ is now:

$$\delta\tilde{q}_a(\omega,\boldsymbol{k}) + 4\pi\alpha_a(\omega,\boldsymbol{k})\sum_b\sum_l I_l(a_{ab})\,\delta\tilde{q}_b(\omega - l\omega_0,\boldsymbol{k}) = \delta\tilde{q}_a^{\,\mathbf{source}}(\omega,\boldsymbol{k}) \tag{43.5}$$

Here $\alpha_a(\omega,\boldsymbol{k})$ is the polarizability of the component a:

$$\alpha_a(\omega,\boldsymbol{k}) = \frac{e_a^2 n_a}{k^2}\int d\boldsymbol{P}\,\frac{\boldsymbol{k}\cdot(\partial F_a/\partial\boldsymbol{P})}{\omega - \boldsymbol{k}\cdot\boldsymbol{V} + i\Delta}$$

$$\varepsilon(\omega,\boldsymbol{k}) = 1 + 4\pi\sum_a \alpha_a(\omega,\boldsymbol{k}) \tag{43.6}$$

and

$$a_{ab} = \left(\frac{e_a}{m_a} - \frac{e_b}{m_b}\right)\frac{\boldsymbol{k}\cdot\boldsymbol{E}}{\omega_0^2}.$$

The set of equations (43.5) is solved by using a perturbation expansion in the small parameter m_e/m_i. To zeroth order, i.e. for $m_i\to\infty$, we find from (43.5)

$$\delta\tilde{q}_e(\omega,\boldsymbol{k}) = \frac{\delta\tilde{q}_e^{\,\mathbf{source}}(\omega,\boldsymbol{k})}{\varepsilon(\omega,\boldsymbol{k})} - \frac{4\pi\alpha_e(\omega,\boldsymbol{k})}{\varepsilon(\omega,\boldsymbol{k})}\sum_n I_n(a_e)\,\delta\tilde{q}_i^{\,\mathbf{source}}(\omega - n\omega_0,\boldsymbol{k})$$

$$\delta\tilde{q}_i(\omega,\boldsymbol{k}) = \delta\tilde{q}_i^{\,\mathbf{source}}(\omega,\boldsymbol{k}). \tag{43.7}$$

The function $\varepsilon(\omega,\boldsymbol{k})$ entering here is defined by (43.6) with $m_i\to\infty$.

We now go back from the functions $\delta\tilde{q}_a(\omega,\boldsymbol{k})$ to the functions $\delta q_a(\omega,\boldsymbol{k})$. To this purpose we use the following relations, which are inverse to (43.4):

$$\delta q_a(\boldsymbol{k},t) = \delta\tilde{q}_a(\boldsymbol{k},t)\,e^{i a_a \sin\omega_0 t} = \sum_n I_n(a_a)\,e^{in\omega_0 t}\,\delta\tilde{q}_a(\boldsymbol{k},t)$$

$$\delta q_a(\omega,\boldsymbol{k}) = \sum_n I_n(a_a)\,\delta\tilde{q}_a(\omega + n\omega_0,\boldsymbol{k}). \tag{43.8}$$

We thus obtain from (43.7):

$$\delta q_e(\omega, \boldsymbol{k}) = \sum_n \sum_m I_n(a_e) I_m(a_e) \left\{ \frac{\delta q_e^{\text{source}}[\omega + (n-m)\omega_0, \boldsymbol{k}]}{\varepsilon(\omega + n\omega_0, \boldsymbol{k})} - \frac{4\pi\alpha_e(\omega + n\omega_0, \boldsymbol{k})}{\varepsilon(\omega + n\omega_0, \boldsymbol{k})} \delta q_i^{\text{source}}[\omega + (n-m)\omega_0, \boldsymbol{k}] \right\}$$

$$\delta q_i(\omega, \boldsymbol{k}) = \delta q_i^{\text{source}}(\omega, \boldsymbol{k}) . \tag{43.9}$$

We substitute these solutions into the right-hand side of Poisson's equation:

$$\delta \boldsymbol{E}(\omega, \boldsymbol{k}) = -\frac{4\pi}{k^2} i\boldsymbol{k}[\delta q_e(\omega, \boldsymbol{k}) + \delta q_i(\omega, \boldsymbol{k})] \tag{43.10}$$

which follows from (43.2), and use the identity

$$\frac{1}{\varepsilon(\omega, \boldsymbol{k})} = 1 - \frac{4\pi\alpha_e}{\varepsilon(\omega, \boldsymbol{k})} . \tag{43.11}$$

As a result we find :

$$\delta \boldsymbol{E}(\omega, \boldsymbol{k}) = \sum_n \sum_m I_n(a_e) I_m(a_e) \frac{\delta \boldsymbol{E}^{\text{source}}[\omega + (n-m)\omega_0, \boldsymbol{k}]}{\varepsilon(\omega + n\omega_0, \boldsymbol{k})} \tag{43.12}$$

Thus, to zeroth order in m_e/m_i, we expressed $\delta\boldsymbol{E}(\omega,\boldsymbol{k})$ through the Fourier components $\delta E^{\text{source}}(\omega,\boldsymbol{k})$. From (43.12), with the use of (43.3) we find and expression for the spatial Fourier transform of the field fluctuations :

$$\delta \boldsymbol{E}(\boldsymbol{k}, t) \int_0^\infty d\tau \left(\frac{1}{\varepsilon}\right)_{\boldsymbol{k},\tau} \exp\left\{-i a_e[\sin\omega_0(t-\tau) - \sin\omega_0 t]\right\} \cdot \delta \boldsymbol{E}^{\text{source}}(\boldsymbol{k}, t-\tau) \tag{43.13}$$

where we used :

$$\frac{1}{\varepsilon(\omega, \boldsymbol{k})} = \int_0^\infty d\tau\, e^{i\omega\tau} \left(\frac{1}{\varepsilon}\right)_{\boldsymbol{k},\tau} . \tag{43.14}$$

Using now the solution (43.13) and Eq (42.11) for $(\delta\boldsymbol{E}\cdot\delta\boldsymbol{E})^{\text{source}}_{\omega,\boldsymbol{k},t}$ we find the spatial spectral density of the field fluctuations :

$$(\delta\boldsymbol{E}\cdot\delta\boldsymbol{E})_{\boldsymbol{k},t} = \frac{(4\pi)^2 e^2}{k^2} \left\{ \int d\boldsymbol{P} \frac{n_e F_e(\boldsymbol{P}, t)}{|\varepsilon(\boldsymbol{k}\cdot\boldsymbol{V}, \boldsymbol{k})|^2} + \int_0^\infty d\tau \int_0^\infty d\tau' \int d\boldsymbol{P} \left(\frac{1}{\varepsilon}\right)_{\boldsymbol{k},\tau} \left(\frac{1}{\varepsilon}\right)^*_{\boldsymbol{k},\tau'} \exp\left\{ i\boldsymbol{k}\cdot\boldsymbol{V}(\tau-\tau') - i a_e[\sin\omega_0(t-\tau) - \sin\omega_0(t-\tau')]\right\} n_i F_i(\boldsymbol{P}) \right\} . \tag{43.15}$$

The first term on the right-hand side represents the contribution of the electrons and the second one the contribution of the ions. The latter still depends on the first time. Averaging Eq. (43.15) over a period $2\pi/\omega_0$ we obtain [33] :

$$(\delta\boldsymbol{E}\cdot\delta\boldsymbol{E})_{\boldsymbol{k},t} = \frac{(4\pi)^2 e^2}{k^2} \left\{ \int d\boldsymbol{P} \frac{n_e F_e(\boldsymbol{P}, t)}{|\varepsilon(\boldsymbol{k}\cdot\boldsymbol{V}, \boldsymbol{k})|^2} + \sum_n I_n^2(a_e) \frac{n_i}{|\varepsilon(n\omega_0, \boldsymbol{k})|^2} \right\} . \tag{43.16}$$

In deriving this equation we used the fact that, for $m_i \to \infty$, the distribution function of the ions $F_i(v) = \delta(v)$.

Equation (43.16) determines, to zeroth order in m_e/m_i, the time-averaged distribution of the energy of the electromagnetic fluctuations in a plasma, in the presence of a high-frequency electric field.

44. KINETIC EQUATION OF A PLASMA IN A HIGH-FREQUENCY ELECTRIC FIELD

We now derive a kinetic equation for the function $F_a(\boldsymbol{P}_a, t)$. The collision integral is defined by Eq. (41.7) in terms of the spatial spectral density of the fluctuations δN_a, $\delta \boldsymbol{E}$. In order to define this spectral density we go back to Eq. (43.1) for the function $\delta N_a(\boldsymbol{k}, \boldsymbol{P}_a - (e_a/\omega_0)\boldsymbol{E}\cos\omega_0 t, t)$. We multiply (43.1) by $\delta \boldsymbol{E}^*(\boldsymbol{k}, t)$ and average: we thus find an expression for the desired function. The latter is represented as a sum of two terms. The first of these is expressed in terms of the spectral density of the field fluctuations: we already know it. The second term is of the form

$$\left(\delta N_a^{\mathbf{source}}\, \delta \boldsymbol{E}\right)_{\boldsymbol{k},\, \boldsymbol{P}_a} - (e_a/\omega_0)\boldsymbol{E}\cos\omega_0 t\,. \tag{44.1}$$

We substitute here the expression of $\delta \boldsymbol{E}(\boldsymbol{k}, t)$ from (43.13), in order to express the function (44.1) in terms of the spectral density of the fluctuations $\delta N_a^{\mathbf{source}}$, $\delta \boldsymbol{E}^{\mathbf{source}}$, which we know. Thus, the determination of the collision integral $I_a(\boldsymbol{P}_a, t)$ reduces to the substitution of known results: we give the result in a moment.

For a spatially homogeneous plasma the kinetic equation for the electron distribution function[†] is

$$\frac{\partial F_e}{\partial t} = I_e(\boldsymbol{P}_e, t) = I_{ee} + I_{ei}\,. \tag{44.2}$$

Here I_{ee} is the electron-electron collision integral:

$$I_{ee} = 2e^4 n \frac{\partial}{\partial P_{e\alpha}} \int d\boldsymbol{k}\, d\boldsymbol{P}'_e \frac{k_\alpha k_\beta}{k^4} \frac{\delta(\boldsymbol{k}\cdot\boldsymbol{V}_e - \boldsymbol{k}\cdot\boldsymbol{V}'_e)}{|\varepsilon(\boldsymbol{k}\cdot\boldsymbol{V}_e, \boldsymbol{k})|^2} \left[\frac{\partial F_e}{\partial P_{e\beta}} F_e - \frac{\partial F_e}{\partial P'_{e\beta}} F_e\right]. \tag{44.3}$$

This expression has the same form as the Balescu-Lenard collision integral. It does not depend explicitly on the electric field. Such a dependence only appears on reverting to the distribution function f_e [see (41.5)].

The electron-ion collision integral, for $m_i \to \infty$, is :

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† **In the approximation $m_i \to \infty$ the ions are motionless.**

$$I_{ei} = \frac{2e^2 e_i^2 n_i}{\pi} \frac{\partial}{\partial P_{e\alpha}} \int_0^\infty d\tau \int_0^\infty d\tau' \int_0^\infty d\tau'' \int d\boldsymbol{k} \frac{k_\alpha k_\beta}{k^4} \left(\frac{1}{\varepsilon}\right)_{\boldsymbol{k},\tau'} \left(\frac{1}{\varepsilon}\right)_{\boldsymbol{k},\tau}$$

$$\exp\left\{-\Delta\tau - i\boldsymbol{k}\cdot\boldsymbol{V}_e\tau + i\,a_e\left[\sin\omega_0(t-\tau') - \sin\omega_0(t-\tau-\tau'')\right]\right\} \cdot \frac{\partial F_e(\boldsymbol{P}_e, t)}{\partial P_{e\beta}} \tag{44.4}$$

Using the expansion (43.3), this equation can be transformed into [33]

$$I_{ei} = 2e^2 e_i^2 n_i \frac{\partial}{\partial P_{e\alpha}} \sum_n \sum_m \int d\boldsymbol{k} \frac{k_\alpha k_\beta}{k^4} I_n(a_e)\, I_m(a_e)$$

$$\frac{\delta(m\omega_0 - \boldsymbol{k}\cdot\boldsymbol{V}_e)}{\varepsilon(m\omega_0, \boldsymbol{k})\,\varepsilon^*(n\omega_0, \boldsymbol{k})} \exp\left[i(n-m)\omega_0 t\right] \frac{\partial F_e(\boldsymbol{P}_e, t)}{\partial P_{e\beta}} \;. \tag{44.5}$$

The expressions (44.4), (44.5) contain all the harmonics of the frequency ω_0. On averaging (44.5) over a period we find

$$I_{ei} = 2e^2 e_i^2 n_i \frac{\partial}{\partial P_{e\alpha}} \sum_n \int d\boldsymbol{k} \frac{k_\alpha k_\beta}{k^4} I_n^2(a_e) \frac{\delta(n\omega_0 - \boldsymbol{k}\cdot\boldsymbol{V}_e)}{|\varepsilon(n\omega_0, \boldsymbol{k})|^2} \frac{\partial F_e(\boldsymbol{P}_e, t)}{\partial P_{e\beta}} \tag{44.6}$$

We recall that this expression was obtained with the assumption $m_i = \infty$; hence it does not depend on the ion distribution function.

We now consider the properties of the collision integral I_e. It is easy to show that

$$n_e \int d\boldsymbol{P}\, \phi(\boldsymbol{P})\, I_e = 0 \quad \text{for} \quad \phi = 1, \boldsymbol{P}\,. \tag{44.7}$$

These properties ensure the conservation of the number of particles and of the momentum. We now consider the energy balance. For an ideal plasma the collision integral does not contribute to the balance equation for the average kinetic energy, hence we obtain from Eq. (41.1)

$$\frac{\partial}{\partial t} \sum_a n_a \int d\boldsymbol{p} \frac{p^2}{2m_a} f_a = \boldsymbol{j}\cdot\boldsymbol{E}\,. \tag{44.8}$$

We go over from the function f_a to the function F_a. Keeping in F_a only the slowly varying part we obtain, for $m_i = \infty$:

$$\frac{\partial}{\partial t} \sum_a n_a \int d\boldsymbol{p} \frac{p^2}{2m_a} f_a = \frac{\partial}{\partial t} n_e \int d\boldsymbol{P} \frac{P^2}{2m_e} F_e \tag{44.9}$$

From this equation it follows, if we use (44.2), that

$$\frac{\partial}{\partial t} n_e \int d\boldsymbol{P} \frac{P^2}{2m_e} F_e = n_e \int d\boldsymbol{P} \frac{P^2}{2m_e} I_{ei}\,. \tag{44.10}$$

Finally, from (44.10) and (44.8) we obtain

$$n_e \int d\boldsymbol{P} \frac{P^2}{2m_e} I_{ei} = \boldsymbol{j}\cdot\boldsymbol{E} = \sigma(\boldsymbol{E}) \frac{E^2}{2} \tag{44.11}$$

which expresses the average energy balance in the plasma in terms of the external electric field.

We calculated the fluctuations of δN_a, $\delta \boldsymbol{E}$ and obtained the corresponding collision integral to zeroth order in the mass ratio m_e/m_i. This approximation is sufficient, in particular for the calculation of the electrical conductivity (see the next section). In many cases, however, such as the calculation of the temperature relaxation, or of the ion-acoustic wave spectrum, this approximation is insufficient.

The method developed here can be used for the extension of the present results to the case of a finite mass ratio: this was done in ref. [48]. We give here as an illustration the expression of the collision integral I_{ei} for external field frequencies $\omega_0 \gg k v_{Te}$:

$$I_{ei} = 2e^2 e_i^2 n_i \frac{\partial}{\partial P_\alpha} \int d\boldsymbol{P}'\, d\boldsymbol{k}\, \frac{k_\alpha k_\beta}{k^4} I_0^2(a_{ei}) \frac{\delta(\boldsymbol{k}\cdot\boldsymbol{V} - \boldsymbol{k}\cdot\boldsymbol{V}')}{|\varepsilon_E(\boldsymbol{k}\cdot\boldsymbol{V}, \boldsymbol{k})|^2}$$

$$\left(\frac{\partial}{\partial P_\beta} - \frac{\partial}{\partial P'_\beta}\right) F_e(\boldsymbol{P}, t)\, F_i(\boldsymbol{P}', t) \qquad (44.12)$$

where

$$\varepsilon_E(\omega, \boldsymbol{k}) = \varepsilon(\omega, \boldsymbol{k}) + [1 - I_0^2(a_{ei})]\, 4\pi\alpha_e \cdot 4\pi\alpha_i\,.$$

The functions $\varepsilon(\omega, \boldsymbol{k}), \alpha_i, \alpha_e, a_{ei}$ were defined in (43.6).

In the absence of the electric field ($E = 0$) Eq. (44.12) reduces to the corresponding Balescu-Lenard collision integral. For $m_e/m_i = 0$, it reduces to (44.6) (with the condition $\omega_0 \gg k v_{eT}$.)

We now use Eq. (44.11) to define the electrical conductivity of the plasma.

45. CONDUCTIVITY OF A PLASMA IN A HIGH-FREQUENCY ELECTRIC FIELD

In order to define the conductivity $\sigma(E)$, we substitute Eq. (44.6) for I_{ei} into (44.11). For the function F_e we take a Maxwell distribution, which is maintained by the electron-electron collisions. Integrating over $\boldsymbol{P}$ in (44.11) we obtain

$$\frac{\sigma(\boldsymbol{E})E^2}{2} = \frac{4(2\pi)^{\frac{1}{2}} e^2 n_e \sum_a e_a^2 n_a}{(k_B T_e)^{\frac{3}{2}}}\, \omega_0^2\, m_e^{\frac{1}{2}}$$

$$\cdot \sum_n \int_0^{k_{\max}} \frac{dk}{k^3} \exp\left(-n^2 \frac{m\omega_0^2}{2k_B T_e k^2}\right) \int_0^1 dx\, \frac{n^2 I_n^2(a_e x)}{|\varepsilon(n\omega_0, \boldsymbol{k})|^2}$$

$$a_e = \frac{e\boldsymbol{k}\cdot\boldsymbol{E}}{m_e \omega_0^2}\,. \qquad (45.1)$$

Let us consider the limiting case of a weak field. From (45.1) it follows that the field may be considered as weak if in the whole range of integration for

k, $|a_e| \ll 1$, i.e.

$$|a_e| = \frac{|e\boldsymbol{k}\cdot\boldsymbol{E}|}{m_e\omega_0^2} \ll 1 \quad \text{for} \quad k_{\mathbf{min}} < k < k_{\mathbf{max}} \tag{45.2}$$

for $\omega_0 \sim \omega_e$, $k_{\mathbf{max}}/k_{\mathbf{min}} \sim 1/\mu$, we have

$$\frac{k_{\mathbf{max}}\, eE}{m_e\omega_e^2} \sim \frac{eEr_{\mathbf{D}}}{k_{\mathbf{B}}T_e}\frac{1}{\mu} \sim \frac{eEl}{k_{\mathbf{B}}T_e}$$

where $l = r_{\mathbf{D}}/\mu$ is the mean-free path for electron-electron collisions. Hence the field is considered weak when the following condition is fulfilled:

$$\frac{eEl}{k_{\mathbf{B}}T_e} \ll 1, \tag{45.3}$$

i.e. the work done by the field over a mean free path must be small compared to $k_{\mathbf{B}}T_e$.

For a weak field, the right-hand side of (45.1) can be expanded in powers of $\boldsymbol{E}$. The series begins with terms in E^2. Cancelling a factor $E^2/2$ on both sides, we find the following expression for the conductivity in a weak field, with account of the plasma polarization:

$$\sigma = \frac{(2\pi)^{\frac{1}{2}}}{3\pi}\frac{\omega_e^2}{\omega_0^2}\frac{e^2}{(k_{\mathbf{B}}T_e)^{\frac{3}{2}}}\frac{\sum\limits_a e_a^2 n_a}{m_e^{\frac{1}{2}}}\int\limits_0^{k_{\mathbf{max}}}\frac{\exp\left(-\dfrac{m_e\omega_0^2}{2k_{\mathbf{B}}T_e k^2}\right)}{k\,|\varepsilon(\omega_0, \boldsymbol{k})|^2} \tag{45.4}$$

This expression agrees with the result of Perel' and Eliashberg [34], in which the conductivity of an equilibrium plasma was calculated.

For $\omega_0 \gg \omega_e$, the effect of the polarization can be neglected. In that case the integration range is cut off by the exponential factor ($k_{\mathbf{min}} \sim \omega_0/v_{Te}$).

In the opposite limiting case, when $\omega_0 \ll \omega_e$, the effect of the polarization is important, and Eq. (45.4) reduces to

$$\sigma = \frac{(2\pi)^{\frac{1}{2}}}{3\pi}\frac{\omega_e^2}{\omega_0^2}\frac{e^2}{(k_{\mathbf{B}}T_e)^{\frac{3}{2}}}\frac{\sum\limits_a e_a^2 n_a}{m_e^{\frac{1}{2}}}\int\limits_0^{k_{\mathbf{max}}} dk\,\frac{1}{k}\,\frac{1}{1 + r_{\mathbf{D}}^2 k^2} \tag{45.5}$$

We now consider the case of a strong field, with $|a_e| \gg 1$. From (45.1) we then obtain

$$\sigma = \frac{8(2\pi)^{\frac{1}{2}}\, e n_e m_e^{\frac{3}{2}}\sum\limits_a e_a^2 n_a}{(k_{\mathbf{B}}T_e)^{\frac{3}{2}}}\frac{\omega_0^4}{E^3}\left\{A(\omega_0) + B(\omega_0)\,\ln\frac{e k_{\mathbf{max}} E}{m_e\omega_0^2}\right\} \tag{45.6}$$

where

$$A(\omega_0) = \sum_n \int_0^{k_{\max}} \frac{dk}{k^4} \frac{\exp -n^2 \dfrac{m_e \omega_0^2}{2 k_B T_e k^2}}{|\varepsilon(n\omega_0, \boldsymbol{k})|^2} n^2 \left\{ \int_0^1 dy\, I_n^2(y) + \int_1^\infty dy \left[I_n^2(y) - \frac{1}{\pi y} \right] + \frac{1}{\pi} \ln \frac{k}{k_{\max}} \right\} \tag{45.7}$$

$$B(\omega_0) = \sum_n \int_0^{k_{\max}} \frac{dk}{k^4} \frac{\exp -n^2 \left(\dfrac{m_e \omega_0^2}{2 k_B T_e k^2} \right)}{|\varepsilon(n\omega_0, \boldsymbol{k})|^2} \frac{n^2}{\pi} . \tag{45.8}$$

For $\omega_0 \ll \omega_e$, the polarization ensures the convergence of these integrals.

For $\omega_0 \gg \omega_e$, the polarization can be neglected. If one also takes into account that

$$A(\omega_0) \ll B(\omega_0) \ln \frac{k_{\max} e E}{m_e \omega_0^2}$$

Eq. (45.6) reduces to [35]:

$$\sigma = \frac{32 \pi^{\frac{1}{2}} \omega_0}{\pi E^3} e n_e \sum_a e_a^2 n_a \left\{ \frac{\pi^{\frac{1}{2}}}{2} \operatorname{erf} x - x e^{-x^2} \right\} \Bigg|_{x = r_{\min}\left(\frac{m_e \omega_0^2}{2k_B T_e}\right)^{\frac{1}{2}}}^{x=\infty} \cdot \ln \frac{k_{\max} e E}{m_e \omega_0^2} . \tag{45.9}$$

We note that the condition of field strength in which (45.6) and (45.9) are valid is

$$\frac{e E r_D}{k_B T_e} \gg 1 .$$

This condition is very strong. The dependence of the conductivity on the field appears already for much weaker fields, when $e E r_D / k_B T_e \ll 1$, but $eEl/k_B T_e \gtrsim 1$. This follows from the criterion (45.2)

For the calculation of the conductivity in a strong field, we assumed that the plasma is stable. In zeroth approximation in the mass ratio m_e/m_i this is indeed true. However, when we consider a finite value for this ratio, the excitation of an aperiodic instability becomes possible (see sect. 8 of ref. [17]). In a non-isothermal plasma, parametric instabilities are possible (see sec. 9, 10 of ref. [17]). The instabilities are excited by perturbations with a wavelength much longer than the Debye radius.

In connection with the existence of instabilities, a question arises. Is the nonlinear field-dependence of the conductivity derived here really observable?

From the previous argument follows that the nonlinear field dependence appears already for relatively weak fields, whenever $e E l / k_B T_e \sim 1$ (l being the

mean free path). This condition can be rewritten in the form $E^2/n k_B T_e \sim \mu^2$ (μ being the plasma parameter). The threshold field for the excitation of the aperiodic instability is known to be defined by $E^2/n k_B T_e \geqslant \mu$ (see Eq. (8,9) in ref. [17]). Thus, for $\mu \ll 1$, the nonlinear field dependence of the conductivity appears before the excitation of the aperiodic instability.

In the non-isothermal plasma, the parametric instability appears after a threshold defined by $E^2/n k_B T_e \geqslant (m_e/m_i)^{\frac{1}{2}} \mu$. For $\mu \ll (m_e/m_i)^{\frac{1}{2}}$, the nonlinear field dependence again appears before the excitation of the instability.

We note that in ref. [48] the temperature relaxation time for a plasma in a strong field was calculated.

46. LOW FREQUENCIES

We consider the kinetic equation for the case of low frequencies, such that

$$\omega_0 \ll \omega_e \sim v_{Te}/r_D \tag{46.1}$$

The ratio of the electron collision frequency to the Langmuir frequency is of the order of the plasma parameter μ; therefore, by Eq. (46.1) the frequency ω_0 is of the order of the collision frequency.

We consider a field of the form

$$E(t) = E \cos \omega_0 t \tag{46.2}$$

this allows us to also consider the case of a constant field. Now, instead of Eqs (42.13), (42.10) we must consider

$$p = P + \frac{e_a E}{\omega_0} \sin \omega_0 t \tag{46.3}$$

$$\int_t^{t-\tau} dt' \int_{-\infty}^{t'} dt'' \, E \cos \omega_0 t'' = \frac{E}{\omega_0^2} \left[\cos \omega_0 t - \cos \omega_0 (t-\tau)\right] . \tag{46.4}$$

Because of (46.1), the right-hand side of (46.4) can be expanded in powers of $\omega_0 \tau$. Retaining only the first two terms in this expansion, we obtain

$$\cos \omega_0 (t-\tau) - \cos \omega_0 t = \omega_0 t \sin \omega_0 t - \frac{1}{2} (\omega_0 \tau)^2 \cos \omega_0 t . \tag{46.5}$$

We now consider the equation for the Fourier components of δN_a. From (43.1), using (46.2) – (46.5) we obtain

$$\delta N_a\left(k, P + \frac{e_a}{\omega_0} E \sin \omega_0 t, t\right) = \delta N_a^{\text{source}}\left(k, P + \frac{e_a}{\omega_0} E \sin \omega_0 t, t\right)$$
$$- e_a n_a \int_0^\infty d\tau \exp\left[-\Delta\tau - i k \cdot \left(V + \frac{e_a}{m_a \omega_0} E \sin \omega_0 t\right) + i a_a(t) \frac{(\omega_0 \tau)^2}{2}\right]$$
$$\cdot \, i E(k, t-\tau) \cdot \frac{\partial f_a\left(P + \frac{e_a}{\omega_0} E \sin \omega_0 (t-\tau), t-\tau\right)}{\partial P} \tag{46.6}$$

We go over to the ordinary variables $\boldsymbol{p}$ (see (46.3)) and note that under the assumption (46.1)

$$\sin \omega_0 (t-\tau) - \sin \omega_0 t = -(\cos \omega_0 t)\, \omega_0 \tau \tag{46.7}$$

Hence, we obtain from (46.6)

$$\delta N_a(\boldsymbol{k}, \boldsymbol{p}, t) = \delta N_a^{\text{source}}(\boldsymbol{k}, \boldsymbol{p}, t)$$
$$- e_a n_a \int_0^\infty d\tau \exp\left[-\Delta\tau - i\boldsymbol{k}\cdot\boldsymbol{v}\tau + i a_a(t) \frac{\omega_0^2 \tau^2}{2}\right]$$
$$\cdot\, \delta\boldsymbol{E}(\boldsymbol{k}, t-\tau) \cdot \frac{\partial f_a(\boldsymbol{p} - e_a \boldsymbol{E}(t)\tau, t-\tau)}{\partial \boldsymbol{p}} \tag{46.8}$$

where

$$a_a(t) = a_a \cos \omega_0 t.$$

In the τ-integral (46.8) the main contribution comes from the region

$$\tau < \tau_{\max} \sim \frac{1}{k_{\max} v_{Te}} \sim \frac{1}{\omega_e}. \tag{46.10}$$

From (46.1):

$$\omega_0 \tau_{\max} \ll 1 \tag{46.11}$$

and thus

$$\tau_{\max} \frac{\partial f_a}{\partial t} \ll f_a. \tag{46.12}$$

It then follows that in Eq (46.8) we may expand the quantities with respect to $\tau(\partial/\partial t)$. In zeroth approximation (ideal plasma) the retardation can be neglected in f_a.

The collision integral I_a is defined by the spectral density $(\delta N_a \delta \boldsymbol{E})_{\omega, \boldsymbol{k}, \boldsymbol{p}}$ in the frequency region $\omega \geqslant \omega_L$ and for $k > r_{\mathrm{D}}^{-1}$. The contribution of the long-range fluctuations to the collision integral will be studied in Chapter 11. As a result, because of (46.1), in the calculation of the spectral density the explicit time-dependence introduced by $\boldsymbol{E}(t)$, $a_a(t)$ into (46.8) can be disregarded. We then find

$$\delta N_a(\omega, \boldsymbol{k}, \boldsymbol{p}) = \delta N_a^{\text{source}}(\omega, \boldsymbol{k}, \boldsymbol{p})$$
$$- e_a n_a \int_0^\infty d\tau \exp\left[-\Delta\tau + i(\omega - \boldsymbol{k}\cdot\boldsymbol{v})\tau + i a_a(t) \frac{\omega_0^2 \tau^2}{2}\right]$$
$$\delta\boldsymbol{E}(\omega, \boldsymbol{k}) \cdot \frac{\partial f_a(\boldsymbol{p} - e_a \boldsymbol{E}(t)\tau, t)}{\partial \boldsymbol{p}}. \tag{46.13}$$

From Eqs (41.2) and (46.13) we obtain an equation for $\delta\boldsymbol{E}(\omega, \boldsymbol{k})$:

$$\varepsilon(\omega, \boldsymbol{k})\, \delta\boldsymbol{E}(\omega, \boldsymbol{k}) = -4\pi i \frac{\boldsymbol{k}}{k^2} \sum_a e_a \int d\boldsymbol{p}\, \delta N_a^{\text{source}}(\omega, \boldsymbol{k}, \boldsymbol{p}) \equiv \delta\boldsymbol{E}^{\text{source}}(\omega, \boldsymbol{k}) \tag{46.14}$$

The dielectric constant appearing here is defined by :

$$\varepsilon(\omega,\boldsymbol{k}) = 1 - i\sum_a \frac{4\pi e_a^2 n_a}{k^2}\int_0^\infty d\tau \int d\boldsymbol{p}$$

$$\cdot \exp\left[-\Delta\tau + i(\omega - \boldsymbol{k}\cdot\boldsymbol{v})\,\tau + i\,a_a(t)\,\frac{\omega_0^2\tau^2}{2}\right]$$

$$\cdot\ \boldsymbol{k}\cdot\frac{\partial f_a\,(\boldsymbol{p}-e_a\boldsymbol{E}(t)\,\tau,t)}{\partial\boldsymbol{p}}\ . \tag{46.15}$$

The expressions for the spectral density of the fluctuations δE^{source} can be obtained from (42.8), using (41.5) and (46.3) – (46.5) :

$$\left(\delta\boldsymbol{E}\cdot\delta\boldsymbol{E}\right)^{\text{source}}_{\omega,\boldsymbol{k}} = \sum_a \frac{(4\pi)^2 e_a^2 n_a}{k^2}\ 2\,\mathrm{Re}\int_0^\infty d\tau\int d\boldsymbol{p}$$

$$\cdot \exp\left[-\Delta\tau + i(\omega - \boldsymbol{k}\cdot\boldsymbol{V})\,\tau + i\,a_a(t)\,\frac{\omega_0^2\tau^2}{2}\right] f_a(\boldsymbol{p}-e_a\boldsymbol{E}(t)\,\tau,t)\ . \tag{46.16}$$

This expression can be rewritten in a more convenient form. We therefore make the change of integration variables

$$\boldsymbol{p} - e_a\boldsymbol{E}(t)\,\tau \rightarrow \boldsymbol{p}\,. \tag{46.17}$$

Using (46.9) we get from (46.16)

$$\left(\delta\boldsymbol{E}\cdot\delta\boldsymbol{E}\right)^{\text{source}}_{\omega,\boldsymbol{k}} = \sum_a \frac{(4\pi)^2 e_a^2 n_a}{k^2}\ 2\int_0^\infty d\tau\int d\boldsymbol{p}$$

$$\cdot \cos\left[(\omega - \boldsymbol{k}\cdot\boldsymbol{v})\,\tau - a_a(t)\,\frac{\omega_0^2\tau^2}{2}\right] f_a(\boldsymbol{p},t)\ . \tag{46.18}$$

From (46.14) follows that the spectral density of field fluctuations is expressed in the usual way through the functions (46.15) and (46.18)

$$(\delta\boldsymbol{E}\cdot\delta\boldsymbol{E})_{\omega,\boldsymbol{k}} = \frac{\left(\delta\boldsymbol{E}\cdot\delta\boldsymbol{E}\right)^{\text{source}}_{\omega,\boldsymbol{k}}}{|\varepsilon(\omega,\boldsymbol{k})|^2}\ . \tag{46.19}$$

We now possess everything that is required for the collision integral

$$I_a(\boldsymbol{p},t) = -\frac{e_a}{n_a}(2\pi)^{-4}\int d\omega\ d\boldsymbol{k}\,\mathrm{Re}\,(\delta N_a\,\delta\boldsymbol{E})_{\omega,\boldsymbol{k},\boldsymbol{p}} \tag{46.20}$$

The result is

$$I_a(\boldsymbol{p}\,,t) = \sum_b \frac{2e_a^2 e_b^2 n_b}{\pi^2} \frac{\partial}{\partial p_i} \int d\omega\, d\boldsymbol{k}\, d\boldsymbol{p}' \frac{k_i k_j}{k^4 |\varepsilon(\omega, \boldsymbol{k})|^2}$$
$$\cdot \operatorname{Re} \int_0^\infty d\tau \exp\left[-\Delta\tau + i(\omega - \boldsymbol{k}\cdot\boldsymbol{v})\,\tau + i\, a_a(t)\, \frac{\omega_0^2 \tau^2}{2}\right]$$
$$\cdot \operatorname{Re} \int_0^\infty d\tau' \exp\left[-\Delta\tau' + i(\omega - \boldsymbol{k}\cdot\boldsymbol{v}')\,\tau' + i\, a_a(t)\, \frac{\omega_0^2 \tau'^2}{2}\right]$$
$$\left(\frac{\partial}{\partial p_j} - \frac{\partial}{\partial p_j'}\right) f_a(\boldsymbol{p} - e_a \boldsymbol{E}(t)\,\tau', t)\, f_b(\boldsymbol{p}' - e_b \boldsymbol{E}(t)\,\tau', t). \tag{46.21}$$

For $\boldsymbol{E}(t) = 0$ this expression reduces to the Balescu-Lenard collision integral (37.1). It is easy to verify that this collision integral possesses the conservation properties :

$$I(t) = \sum_a n_a \int d\boldsymbol{p}\; \phi_a(\boldsymbol{p})\, I_a(\boldsymbol{p}\,,t) = 0 \quad \text{for} \quad \phi_a = 1, \boldsymbol{p}, \frac{p^2}{2m_a}. \tag{46.22}$$

The Balescu-Lenard collision integral and its generalization (46.21) were obtained to the first order of approximation in the plasma parameter μ. As the dynamical polarization is properly taken into account, these integrals converge for small values of k (i.e. for large distances).

The contribution of the short-distance interactions is, however, not completely correct: this results in a logarithmic divergence for large k. Because of the logarithmic character this divergence is not important for an ideal plasma, but for a nonideal plasma the short-rage interactions are important in the determination of the non-dissipative properties. As a result, we need an expression for the collision integral which should describe with sufficient accuracy the behaviour at both large and small distances. It will be seen in sections 55 and 56 that this problem can be solved by using an averaged dynamical polarization instead of the exact one. This amounts to introducing an effective potential.

47. THE EFFECTIVE POTENTIAL ORIGINATING FROM THE AVERAGE DYNAMICAL POLARIZATION

From the expression for the function $\operatorname{Re}(\delta N_a\, \delta E)_{\omega.\,\boldsymbol{k},\boldsymbol{p}}$ defining the Balescu-Lenard collision integral (37.1) it is clear that, due to the polarization, the Fourier transform of the interaction potential is renormalized as follows :

$$\nu_{ab}(\boldsymbol{k}) \to \frac{\nu_{ab}(\boldsymbol{k})}{|\varepsilon(\boldsymbol{k}\cdot\boldsymbol{v}\,,\, \boldsymbol{k})|^2}\,, \quad \nu_{ab}(\boldsymbol{k}) = \frac{4\pi e_a n_a}{k^2} \tag{47.1}$$

On this basis, the effective potential taking into account the dynamical polarization, averaged over the velocities, can be defined as follows [7] :

$$\tilde{\nu}_{ab}(\boldsymbol{k}) = \nu_{ab}(\boldsymbol{k}) \frac{1}{\sum_d e_d^2 n_d} \sum_c e_c^2 n_c \int d\boldsymbol{p} \frac{f_c(\boldsymbol{p}, t)}{|\varepsilon(\boldsymbol{k}\cdot\boldsymbol{v}, \boldsymbol{k})|^2} . \tag{47.2}$$

Let us consider this expression in the local-equilibrium approximation, when f_c is a Maxwellian with $T_c = T$. Using Eq. (36.19) and the definition of the Debye radius, we obtain from (47.2) the following result

$$\tilde{\nu}_{ab}(k) = \nu_{ab}(k) \frac{r_D^2 k^2}{1 + r_D^2 k^2} \tag{47.3}$$

Hence, using (36.23) we find

$$\tilde{\Phi}_{ab}(r) = \Phi_{ab}(r)\, e^{-r/r_D} , \quad \Phi_{ab}(r) = \frac{e_a e_b}{r} . \tag{47.4}$$

Hence, in local equilibrium the effective potential reduces to the Debye potential.

The effective potential can be expressed in terms of the spectral density of the field fluctuations, $(\delta\boldsymbol{E}\cdot\delta\boldsymbol{E})_{\boldsymbol{k}}$. Using Eq. (35.7)

$$\tilde{\nu}_{ab}(k) = \frac{e_a e_b}{\sum_c e_c^2 n_c} \frac{1}{4\pi} (\delta\boldsymbol{E}\cdot\delta\boldsymbol{E})_{\boldsymbol{k}} = \nu_{ab}(k) \frac{(\delta q\, \delta q)_{\boldsymbol{k}}}{\sum_c e_c^2 n_c} \tag{47.5}$$

where $(\delta q\ \delta q)_{\boldsymbol{k}}$ is the spectral density of the charge fluctuations.

Equation (47.5) can be taken as a definition of the effective potential to be used in more general cases, as for instance, in the presence of an external field. We rewrite it as

$$\tilde{\nu}_{ab}(\boldsymbol{k}) = \nu_{ab}(\boldsymbol{k}) \frac{1}{2\pi \sum_c e_c^2 n_c} \int d\omega \frac{(\delta q\ \delta q)^{\mathbf{source}}_{\omega, \boldsymbol{k}}}{|\varepsilon(\omega, \boldsymbol{k})|^2} \tag{47.6}$$

where we use the fact that

$$(\delta q\ \delta q)^{\mathbf{source}}_{\omega, \boldsymbol{k}} = \frac{k^2}{(4\pi)^2} (\delta\boldsymbol{E}\cdot\delta\boldsymbol{E})^{\mathbf{source}}_{\omega, \boldsymbol{k}} . \tag{47.7}$$

We now show that for an ideal gas, i.e., for $g_{ab} = 0$, the effective potential reduces to the interaction potential. We note that from the definition of δq:

$$(\delta q\ \delta q)_{\boldsymbol{k}} = \sum_a \sum_b e_a e_b \int d\boldsymbol{p}_1\, d\boldsymbol{p}_2\, (\delta N_a\ \delta N_b)_{\boldsymbol{k}, \boldsymbol{p}_1, \boldsymbol{p}_2} .$$

From (26.11) we find for the ideal gas

$$(\delta N_a\ \delta N_b)_{\boldsymbol{k}, \boldsymbol{p}_1, \boldsymbol{p}_2} = \delta_{ab}\ \delta(\boldsymbol{p} - \boldsymbol{p}')\ n_a f_a . \tag{47.8}$$

From (47.7), (47.8) we find, for $g_{ab} = 0$:

$$(\delta q\ \delta q)_{\boldsymbol{k}} = \sum_a e_a^2 n_a \ . \tag{47.9}$$

Combining this result with (47.5), we find that for $g_{ab}=0$, $\tilde{\Phi}_{ab} = \Phi_{ab}$.

Comparing Eqs (47.1) and (47.6) we see that the introduction of the effective potential corresponds to the following substitution in the collision integral:

$$\frac{\nu_{ab}(\boldsymbol{k})}{|\varepsilon(\omega,\boldsymbol{k})|^2} \to \nu_{ab}(\boldsymbol{k}) \frac{1}{2\pi \sum_c e_c^2 n_c} \int d\omega \frac{(\delta q\ \delta q)^{\text{source}}_{\omega,\boldsymbol{k}}}{|\varepsilon(\omega,\boldsymbol{k})|^2} \equiv \nu_{ab}(\boldsymbol{k}) \left\langle \frac{1}{|\varepsilon(\omega,\boldsymbol{k})|^2} \right\rangle^{(\omega)} \tag{47.10}$$

i.e., it amounts to averaging the function $1/|\varepsilon(\omega,\boldsymbol{k})|^2$ over ω with the distribution function

$$f(\omega) = \frac{1}{\sum_c e_c^2 n_c} (\delta q\ \delta q)^{\text{source}}_{\omega,\boldsymbol{k}} \ . \tag{47.11}$$

Using this result, we introduce the effective potential into the collision integral (46.21) by the prescription (47.10) and then perform the integrations over ω and over τ':

$$I_a(\boldsymbol{p},t) = \sum_b \frac{2 e_a^2 e_b^2 n_b}{\pi} \frac{\partial}{\partial p_i} \int d\boldsymbol{k} \int d\boldsymbol{p}' \int_0^\infty d\tau \frac{k_i k_j}{k^4}$$

$$\cdot \left\langle \frac{1}{|\varepsilon(\omega,\boldsymbol{k})|^2} \right\rangle^{(\omega)} \cos\left[(\boldsymbol{k}\cdot\boldsymbol{v} - \boldsymbol{k}\cdot\boldsymbol{v}')\tau - a_{ab}(t) \frac{\omega_0^2 \tau^2}{2} \right]$$

$$\left(\frac{\partial}{\partial p_j} - \frac{\partial}{\partial p_j'} \right) f_a(\boldsymbol{p} - e_a \boldsymbol{E}(t)\tau)\, f_b(\boldsymbol{p}' - e_b \boldsymbol{E}(t)\tau)$$

$$a_{ab}(t) = \left(\frac{e_a}{m_a} - \frac{e_b}{m_b} \right) \frac{\boldsymbol{k}\cdot\boldsymbol{E}(t)}{\omega_0^2} \ . \tag{47.13}$$

The approximate expression (47.12) for the collision integral will be used for the calculation of the conductivity of the plasma.

48. ELECTRICAL CONDUCTIVITY AND EFFECTIVE POTENTIAL

We consider the balance equation for the electrical current density $\boldsymbol{j} = \sum_a e_a n_a \int d\boldsymbol{p}\, \boldsymbol{v} f_a$. From the kinetic equation (41.1) follows:

$$\frac{\partial \boldsymbol{j}}{\partial t} = \sum_a \frac{e_a^2 n_a}{m_a} \boldsymbol{E}(t) + \sum_a e_a n_a \int d\boldsymbol{p}\, \boldsymbol{v}\, I_a \ . \tag{48.1}$$

We substitute (47.12) into the second term on the right-hand side, integrate by parts over $\boldsymbol{p}$, symmetrize with respect to $a, \boldsymbol{p}$; $b, \boldsymbol{p}'$ and make the substitution $\boldsymbol{p} - e_a \boldsymbol{E}(t)\tau \to \boldsymbol{p}$:

$$\sum_a e_a n_a \int dp\, \boldsymbol{v}\, I_a = -\sum_a \sum_b \frac{e_a^2 e_b^2 n_a n_b}{\pi} \left(\frac{e_a}{m_a} - \frac{e_b}{m_b}\right)$$

$$\int_0^\infty d\tau \int d\boldsymbol{k}\, d\boldsymbol{p}\, d\boldsymbol{p}' \, \frac{\boldsymbol{k}}{k^4} \left\langle \frac{1}{|\varepsilon(\omega, \boldsymbol{k})|^2} \right\rangle^{(\omega)} \cos\left[(\boldsymbol{k}\cdot\boldsymbol{v} - \boldsymbol{k}\cdot\boldsymbol{v}')\tau + a_{ab}(t)\frac{\omega_0^2\tau^2}{2}\right]$$

$$\boldsymbol{k}\cdot\left(\frac{\partial}{\partial \boldsymbol{p}} - \frac{\partial}{\partial \boldsymbol{p}'}\right) f_a f_b \, . \tag{48.2}$$

We continue the calculation with the function :

$$f_a = A \exp\left[-\frac{m_a(\boldsymbol{v} - \boldsymbol{u}_a)^2}{2 k_{\mathrm{B}} T}\right] . \tag{48.3}$$

Then,

$$\boldsymbol{k}\cdot\left(\frac{\partial}{\partial \boldsymbol{p}} - \frac{\partial}{\partial \boldsymbol{p}'}\right) f_a f_b = -\frac{1}{k_{\mathrm{B}}T}\left[(\boldsymbol{k}\cdot\boldsymbol{v} - \boldsymbol{k}\cdot\boldsymbol{v}') - (\boldsymbol{k}\cdot\boldsymbol{u}_a - \boldsymbol{k}\cdot\boldsymbol{u}_b)\right] f_a f_b . \tag{48.4}$$

(1) We consider the result of the substitution of the first term of (48.4) into (48.2). Noting that

$$\int_0^\infty d\left[\boldsymbol{k}\cdot(\boldsymbol{v} - \boldsymbol{v}')\tau + a_{ab}(t)\frac{\omega_0^2\tau^2}{2}\right] \cos\left[\boldsymbol{k}\cdot(\boldsymbol{v} - \boldsymbol{v}')\tau + a_{ab}(t)\frac{\omega_0^2\tau^2}{2}\right] = 0$$

we transform the first part of (48.2) into

$$\left[\sum_a e_a n_a \int d\boldsymbol{p}\, \boldsymbol{v}\, I_a\right]_1 = \alpha \boldsymbol{E} \cos \omega_0 t \tag{48.5}$$

with

$$\alpha = -\sum_a \sum_b \frac{e_a^2 e_b^2 n_a n_b}{k_{\mathrm{B}} T}\left(\frac{e_a}{m_a} - \frac{e_b}{m_b}\right) I_1^{ab} \tag*{$(48.6)_1$}$$

and

$$I_1^{ab} = \frac{1}{\pi}\int_0^\infty d\tau\, \tau \int d\boldsymbol{k}\, d\boldsymbol{p}\, d\boldsymbol{p}' \, \frac{[\boldsymbol{k}\cdot\boldsymbol{E}(t)]^2}{k^4 E^2(t)} \left\langle \frac{1}{|\varepsilon(\omega, \boldsymbol{k})|^2}\right\rangle^{(\omega)}$$

$$\cdot \cos\left[(\boldsymbol{k}\cdot\boldsymbol{v} - \boldsymbol{k}\cdot\boldsymbol{v}')\tau + a_{ab}(t)\frac{\omega_0^2\tau^2}{2}\right] f_a(\boldsymbol{p}) f_b(\boldsymbol{p}') . \tag*{$(48.6)_2$}$$

(2) Substitution of the second term of (48.4) into (48.2) leads to

$$\left(\sum_a e_a n_a \int d\boldsymbol{p}\, \boldsymbol{v}\, I_a\right)_2 = -\nu \boldsymbol{j} \tag{48.7}$$

where

$$\nu = \sum_a \sum_b \frac{2 e_a^2 e_b^2 n_a n_b}{k_{\mathrm{B}} T (e_a n_a - e_b n_b)}\left(\frac{e_a}{m_a} - \frac{e_b}{m_b}\right) I_2^{ab} \tag{48.8}$$

and

$$I_2^{ab} = \frac{1}{\pi}\int_0^\infty d\tau \int dk\, dp\, dp'\ \frac{(k\cdot j)^2}{k^4 j^2}\left\langle\frac{1}{|\varepsilon(\omega, k)|^2}\right\rangle^{(\omega)}$$

$$\cos\left[(k\cdot v - k\cdot v')\,\tau + a_{ab}(t)\,\frac{\omega_0^2\tau^2}{2}\right] f_a f_b\ . \tag{48.9}$$

Here we used the fact that $j = \frac{1}{2}(e_a n_a - e_b n_b)(u_a - u_b)$.

The expression (48.5), added to the first term on the right-hand side of (48.1) defines the effective field in the plasma in presence of a strong external electric field (with $\omega_0 \ll \omega_L$) and taking account of the average dynamical polarization.

The second term (48.7) describes the dissipation in Eq. (48.1) and therefore defines the conductivity of the plasma under the same conditions.

We consider these results in the case of the weak field, but with due account of the polarization. In that limit, Eqs $(48.6)_2$ and (48.9) yield :

$$I_1^{ab} = \frac{2\pi}{3}\frac{\mu_{ei}}{k_B T}\, r_D\ , \qquad \mu_{ei} = \frac{m_e m_i}{m_e + m_i} \tag{48.10}$$

$$I_2^{ab} = \frac{4\pi}{3}\left(\frac{\mu_{ei}}{2\pi k_B T}\right)^{\frac{1}{2}} \int_0^{k_{max}} \frac{dk}{k}\left\langle\frac{1}{|\varepsilon(\omega, k)|^2}\right\rangle^{(\omega)} \tag{48.11}$$

From (48.10) and $(48.6)_1$ we find

$$a = -\frac{1}{6}\,\frac{e^2}{r_D}\,\frac{e^2 n}{\mu_{ei}}\,k_B T \tag{48.12}$$

and, adding (48.5) to the first term on the right-hand side of (48.1) we find the effective field [41, 49]:

$$E_{eff} = \left(1 - \frac{1}{6}\,\frac{e^2}{r_D k_B T}\right) E\ . \tag{48.13}$$

Thus, the difference between the effective and the external field is proportional to the plasma parameter $\mu \sim e^2/r_D k_B T$.

We now consider Eq. (48.8) for the damping rate ν of the current. In the weak field limit, (47.13) reduces to

$$\left\langle\frac{1}{|\varepsilon(\omega, k)|^2}\right\rangle^{(\omega)} = \frac{1}{\sum_c e_c^2 n_c}\ \sum_a e_a^2 n_a \int dp\, \frac{f_a}{|\varepsilon(k\cdot v, k)|^2} \tag{48.14}$$

As f_a is Maxwellian, we may use the result (36.19). Thus (48.11) becomes

$$I_d = \frac{4\pi}{3}\left(\frac{\mu_{ei}}{2\pi k_B T}\right)^{\frac{1}{2}} \int_0^{k_{max}} dk\ \frac{k r_D}{1 + k^2 r_D^2} = \frac{2(2\pi)^{\frac{1}{2}}}{3}\left(\frac{\mu_{ei}}{k_B T}\right)^{\frac{1}{2}} \ln\left(1 + r_D^2 k_{max}^2\right)^{\frac{1}{2}}\ . \tag{48.15}$$

We substitute this result into (48.8) and obtain

$$\nu = \frac{4(2\pi)^{\frac{1}{2}}}{3} \frac{e^4 n}{\mu_{ei}^{\frac{1}{2}}(k_B T)^{\frac{3}{2}}} \ln\left(1 + r_D^2 k_{\max}^2\right)^{\frac{1}{2}} . \tag{48.16}$$

From (48.13) we see that, for $e_i = |e_e|$, the effective field is independent of the index a, hence (48.1) combined with (48.13), (48.7) becomes

$$\frac{\partial \boldsymbol{j}}{\partial t} + \nu \boldsymbol{j} = \frac{e^2 n}{\mu_{ei}} \boldsymbol{E}_{\text{eff}} . \tag{48.17}$$

It then follows that

$$\boldsymbol{j}(\omega) = \frac{e^2 n}{\mu_{ei}} \frac{1}{-i\omega + \nu} \boldsymbol{E}_{\text{eff}}(\omega) \equiv \sigma(\omega) \boldsymbol{E}_{\text{eff}}(\omega) \tag{48.18}$$

and the real part of the conductivity becomes [7]:

$$\text{Re } \sigma(\omega) = \frac{e^2 n}{\mu_{ei}} \frac{\nu}{\omega^2 + \nu^2} \tag{48.19}$$

with ν given by (48.16).

Had we used the Landau collision integral, we would have found in the expression of ν:

$$\ln\left(1 + r_D^2 k_{\max}^2\right)^{\frac{1}{2}} \equiv L' \longrightarrow \ln r_D k_{\max} \equiv L \tag{48.20}$$

where L is the so-called Coulomb logarithm. From this equation

$$L' - L = \begin{cases} \frac{1}{2} r_D^2 k_{\max}^2 & \text{for} \quad r_D k_{\max} \ll 1 \\ 1 - r_D k_{\max} & \text{for} \quad 1 - r_D k_{\max} \ll 1 \end{cases} . \tag{48.21}$$

In a static field $(\omega_0 = 0)$ the substitution of L for L' leads to a decrease of the conductivity, as

$$\sigma(\omega = 0) = \frac{e^2 n}{\mu_{ei} \nu} \sim \frac{1}{L'} \tag{48.22}$$

and $L' > L$.

49. FLUCTUATIONS IN THE PRESENCE OF A MAGNETIC FIELD

We consider the influence of an external magnetic field on the spectral density of fluctuations in a non-relativistic plasma. In that case the transverse electromagnetic field created by the particles is small. Therefore the description of the fluctuations, to the approximation of the second moments can start with Eqs (27.7) – (27.10) in which we set

$$\boldsymbol{F}_{a0} = \frac{e_a}{c} (\boldsymbol{v} \times \boldsymbol{B}) . \tag{49.1}$$

Thus, the basic equations are :

$$\left(\frac{\partial}{\partial t} + \boldsymbol{v}\cdot\frac{\partial}{\partial \boldsymbol{r}} + \frac{e_a}{c}(\boldsymbol{v}\times\boldsymbol{B})\cdot\frac{\partial}{\partial \boldsymbol{p}}\right)\left(\delta N_a - \delta N_a^{\text{source}}\right) = -e_a \boldsymbol{E}\cdot\frac{\partial n_a f_a}{\partial \boldsymbol{p}}$$

$$\nabla\times\delta\boldsymbol{E} = 0\,, \quad \nabla\cdot\delta E = 4\pi\sum_a e_a\int d\boldsymbol{p}\,\delta N_a \tag{49.2}$$

$$\left(\frac{\partial}{\partial t} + \boldsymbol{v}\cdot\frac{\partial}{\partial \boldsymbol{r}} + \frac{e_a}{c}(\boldsymbol{v}\times\boldsymbol{B})\cdot\frac{\partial}{\partial \boldsymbol{p}}\right)\left\langle \delta N_a\ \delta N_b\right\rangle^{\text{source}}_{x,t,x',t'} = 0 \tag{49.3}$$

$$\left\langle \delta N_a\ \delta N_b\right\rangle^{\text{source}}_{x,t,x',t'}\Bigg|_{t=t'} = \delta_{ab}\,\delta(x-x')\,n_a f_a(x',t')\,. \tag{49.4}$$

In equation (49.2) the term

$$e_a(\boldsymbol{v}\times\delta\boldsymbol{B})\cdot\frac{\partial f_a}{\partial \boldsymbol{p}} \tag{49.5}$$

is absent. This happens whenever f_a depends only on the components of the momentum which are longitudinal or transverse with respect to the vector $\boldsymbol{B}$, i.e., f_a is independent of the angle variable in momentum space. We will only consider here such distribution functions :

$$f_a(\boldsymbol{p},t) = f_a(p^{\parallel}, p^{\perp}, t)\,. \tag{49.6}$$

In that case the quantity (49.5) vanishes and the equation for f_a is:

$$\frac{\partial f_a}{\partial t} = -\frac{e_a}{n_a}\frac{1}{(2\pi)^3}\frac{\partial}{\partial \boldsymbol{p}}\cdot\int d\boldsymbol{k}\ \mathrm{Re}\,(\delta N_a\delta E)_{\boldsymbol{k},\boldsymbol{p},t}\,. \tag{49.7}$$

It follows that the function (49.6) is slowly varying in time. Thus, for an ideal plasma, the retardation of the function f_a can be neglected and the fluctuation process, under the assumption (49.6), is stationary.

In the case when the distribution function depends on the phase, the fluctuation process is no longer stationary. The calculation of the spectral densities must then be done in analogy with the methods developed in sections 41–46 for the problem of a strong, high-frequency electric field.

We now solve Eqs (49.2), (49.3). Consider first the characteristic equations:

$$\frac{d\boldsymbol{p}}{dt} = \frac{e_a}{c}(\boldsymbol{v}\times\boldsymbol{B})\quad,\quad \frac{d\boldsymbol{r}}{dt} = \boldsymbol{v}\,. \tag{49.8}$$

We denote the initial values (at time t') of the position and momentum by $\boldsymbol{R}\cdot\boldsymbol{P}$. The solution of (49.8) is then :

$$\boldsymbol{p} = (\boldsymbol{P}\cdot\boldsymbol{b})\boldsymbol{b} + [(\boldsymbol{b}\times\boldsymbol{P})\times\boldsymbol{b}]\cos\Omega_a(t-t') + [\boldsymbol{P}\times\boldsymbol{b}]\sin\Omega_a(t-t') \tag{49.9}$$

$$\boldsymbol{r} = \boldsymbol{R} + \frac{(\boldsymbol{b}\cdot\boldsymbol{P})\boldsymbol{b}}{m_a}(t-t') + \frac{[(\boldsymbol{b}\times\boldsymbol{P})\times\boldsymbol{b}]}{m_a\Omega_a}\sin\Omega_a(t-t') + \frac{(\boldsymbol{P}\times\boldsymbol{b})}{m_a\Omega_a}[1-\cos\Omega_a(t-t')] \tag{49.10}$$

with the notation

$$\boldsymbol{b} = \frac{\boldsymbol{B}}{B} \ , \ \Omega_a = \frac{e_a B}{m_a c} \ . \tag{49.11}$$

We also have the inverse relations

$$\boldsymbol{P} = (\boldsymbol{b}\cdot\boldsymbol{p})\boldsymbol{b} + [(\boldsymbol{b}\times\boldsymbol{p})\times\boldsymbol{b}]\cos\Omega_a(t'-t) - (\boldsymbol{b}\times\boldsymbol{p})\sin\Omega_a(t'-t) \tag{49.12}$$

$$\boldsymbol{R} = \boldsymbol{r} + \frac{(\boldsymbol{b}\cdot\boldsymbol{p})\boldsymbol{b}}{m_a}(t'-t) + \frac{[(\boldsymbol{b}\times\boldsymbol{p})\times\boldsymbol{b}]}{m_a\Omega_a}\sin\Omega_a(t'-t)$$
$$- \frac{(\boldsymbol{b}\times\boldsymbol{p})}{\Omega_a}[1-\cos\Omega_a(t'-t)] \ . \tag{49.13}$$

Along with the vectorial expressions, we shall also need the separate components. Taking the vector $\boldsymbol{B}$ parallel to the z-axis we find from (49.9)

$$p_x = P_x\cos\Omega_a(t-t') + P_y\sin\Omega_a(t-t')$$
$$p_y = P_y\cos\Omega_a(t-t') - P_x\sin\Omega_a(t-t')$$
$$p_z = P_z \tag{49.14}$$

and from (49.10)

$$r_x = R_x + \frac{P_x}{m_a\Omega_a}\sin\Omega_a(t-t') + \frac{P_y}{m_a\Omega_a}[1-\cos\Omega_a(t-t')]$$
$$r_y = R_y + \frac{P_y}{m_a\Omega_a}\sin\Omega_a(t-t') - \frac{P_x}{m_a\Omega_a}[1-\cos\Omega_a(t-t')]$$
$$r_z = R_z + \frac{P_z}{m_a}(t-t') \ . \tag{49.15}$$

Using the solution (49.12), (49.13), we find from (49.2)

$$\delta N_a(\omega,\boldsymbol{k},\boldsymbol{p}) = \delta N_a^{\text{source}}(\omega,\boldsymbol{k},\boldsymbol{p})$$
$$- e_a n_a \int_0^\infty d\tau \exp\left\{-\Delta\tau + i[\omega\tau + \boldsymbol{k}\cdot(\boldsymbol{R}(-\tau)-\boldsymbol{r})]\right\}$$
$$\cdot\ \delta\boldsymbol{E}(\omega,\boldsymbol{k})\cdot\frac{\partial}{\partial\boldsymbol{p}} f_a(\boldsymbol{P}(-\tau),t) \ . \tag{49.16}$$

Here we used the fact that f_a is slowly varying in time. For $\boldsymbol{B}=0$ this expression reduces to (35.2). From Eq. (49.2) for $\delta\boldsymbol{E}$ we find

$$\delta\boldsymbol{E}(\omega,\boldsymbol{k}) = -\frac{i\boldsymbol{k}}{k^2}\sum_a 4\pi e_a\int d\boldsymbol{p}\ \delta N_a(\omega,\boldsymbol{k},\boldsymbol{p}) \ . \tag{49.17}$$

We substitute now the expression (49.16) and find :

$$\varepsilon(\omega,\boldsymbol{k})\ \delta E(\omega,\boldsymbol{k}) = -\frac{i\boldsymbol{k}}{k^2}4\pi\sum_a e_a\int d\boldsymbol{p}\ \delta N_a^{\text{source}}(\omega,\boldsymbol{k},\boldsymbol{p}) \equiv \delta E^{\text{source}}(\omega,\boldsymbol{k}). \tag{49.18}$$

We introduced here the dielectric constant of the plasma in the magnetic field :

$$\varepsilon(\omega,\boldsymbol{k}) = 1 - i\sum_a \frac{4\pi e_a^2 n_a}{k^2}\int_0^\infty d\tau\int d\boldsymbol{p}\ \exp\left\{-\Delta\tau + i\left[\omega\tau + \boldsymbol{k}\cdot(\boldsymbol{R}(-\tau)-\boldsymbol{r})\right]\right\}$$
$$\boldsymbol{k}\cdot\frac{\partial f_a(\boldsymbol{P}(-\tau),t)}{\partial\boldsymbol{p}}. \tag{49.19}$$

We transform this equation using the condition (49.6) which implies that $f_a(\boldsymbol{P}(-\tau)) = f_a(\boldsymbol{p})$. We also use the relation :

$$\boldsymbol{k}\cdot\frac{\partial f_a(\boldsymbol{P}(-\tau),t)}{\partial\boldsymbol{p}} = \frac{\partial f_a(\boldsymbol{p})}{\partial p_i}\,k_j\,\frac{\partial p_i}{\partial P_j} = \boldsymbol{P}(-\tau,\boldsymbol{k})\cdot\frac{\partial f_a(\boldsymbol{p})}{\partial\boldsymbol{p}} \tag{49.20}$$

where the vector $\boldsymbol{P}(-\tau,\boldsymbol{k})$ is defined by Eq. (49.12) in which the vector $\boldsymbol{p}$ is replaced by $\boldsymbol{k}$. We introduce a cylindrical reference frame, $p^{\parallel}$, $p^{\perp}$, ϕ with the z-axis along the vector $\boldsymbol{B}$:

$$p_x = p^{\perp}\cos\phi\ ,\quad k_x = k^{\perp}\cos\psi$$
$$p_y = p^{\perp}\sin\phi\ ,\quad k_y = k^{\perp}\sin\psi\,. \tag{49.21}$$

We then obtain from (49.13)

$$\boldsymbol{k}\cdot\boldsymbol{R}(-\tau) = \boldsymbol{k}\cdot\boldsymbol{r} - k^{\parallel}v^{\parallel}\tau - \frac{k^{\perp}v^{\perp}}{\Omega_a}\left[\sin(\phi-\psi+\Omega_a\tau) - \sin(\phi-\psi)\right] \tag{49.22}$$

$$\boldsymbol{P}(-\tau,\boldsymbol{k})\cdot\frac{\partial f_a}{\partial\boldsymbol{p}} = k^{\parallel}\frac{\partial f_a}{\partial p^{\parallel}} + \cos(\phi-\psi+\Omega_a\tau)\,k^{\perp}\frac{\partial f_a}{\partial p^{\perp}}\,. \tag{49.23}$$

In this equation we used the fact that f_a is independent of ϕ.

We now substitute Eqs (49.20), (49.22), (49.23) into (49.19) and use the expansion

$$e^{ix\sin(\phi-\psi)} = \sum_{n=-\infty}^{\infty} I_n(x)\,e^{in(\phi-\psi)}\ ,\quad x = \frac{k^{\perp}v^{\perp}}{\Omega_a}\,. \tag{49.24}$$

As the distribution function is independent of ϕ, we may integrate over this variable, by using the relations :

$$\frac{1}{2\pi}\sum_n\sum_m\int_0^{2\pi} d\phi\ I_n(x)\,I_m(x)\exp\left[-in(\phi-\psi+\Omega_a\tau) + im(\phi-\psi)\right]$$
$$= \sum_m I_m^2(x)\exp(-im\,\Omega_a\tau) \tag{49.25}$$

$$\frac{1}{2\pi}\sum_n\sum_m \int_0^{2\pi} d\phi\; I_n(x)\, I_m(x) \exp\left[-in\,(\phi-\psi+\Omega_a\tau) + im\,(\phi-\psi)\right]$$

$$\cdot \cos\,(\phi-\psi+\Omega_a\,\tau) = \sum_m \frac{m}{x}\, I_m^2(x)\, \exp\,(-im\,\Omega_a\,\tau)\,. \tag{49.25}$$

We also note the relation

$$-i\int_0^\infty d\tau\, \exp\left[-\Delta\tau + i\,(\omega - m\Omega_a - k^{\parallel} v^{\parallel})\,\tau\right] = \frac{1}{\omega - m\Omega_a - k^{\parallel} v^{\parallel} + i\Delta}.$$

We thus obtain the following expression for the dielectric constant

$$\varepsilon(\omega,\, \boldsymbol{k}) = 1 + \sum_a \frac{4\pi e_a^2 n_a}{k^2} \sum_m \int dp^{\perp}\, 2\pi p^{\perp} \int dp^{\parallel}\, I_m^2\left(\frac{k^{\perp} p^{\perp}}{m_a\Omega_a}\right)$$

$$\cdot \frac{1}{\omega - k^{\parallel} v^{\parallel} - m\Omega_a + i\Delta}\left[k^{\parallel}\frac{\partial}{\partial p^{\parallel}} + \frac{m\Omega_a}{v^{\perp}}\frac{\partial}{\partial p^{\perp}}\right] f_a(p^{\parallel}, p^{\perp}, t)\,. \tag{49.26}$$

Under the condition (49.6), the expressions (49.19) and (49.26) are equivalent.

We now study the spectral density of the field fluctuations. Going back to (49.17) we find:

$$\left(\delta\boldsymbol{E}\cdot\delta\boldsymbol{E}\right)_{\omega,\boldsymbol{k}} = \frac{1}{|\varepsilon(\omega,\boldsymbol{k})|^2}\sum_a\sum_b \frac{(4\pi)^2 e_a e_b}{k^2}\int d\boldsymbol{p}\, d\boldsymbol{p}'\left(\delta N_a\,\delta N_b\right)^{\text{source}}_{\omega,\boldsymbol{k},\boldsymbol{p},\boldsymbol{p}'}$$

$$\equiv \frac{\left(\delta\boldsymbol{E}\cdot\delta\boldsymbol{E}\right)^{\text{source}}_{\omega,\boldsymbol{k}}}{|\varepsilon(\omega,\boldsymbol{k})|^2}\,. \tag{49.27}$$

In order to define the spectral density of $\delta N_a^{\text{source}}$ we consider Eq. (49.3). Using the slow time variation of f_a, we find, to zeroth order in the retardation:

$$\left(\delta N_a\,\delta N_b\right)^{\text{source}}_{\omega,\boldsymbol{k},\boldsymbol{p},\boldsymbol{p}'}$$

$$= 2\,\delta_{ab} n_a\, \mathrm{Re}\int_0^\infty d\tau\, \exp\left\{-\Delta\tau + i\left[\omega\tau + \boldsymbol{k}\cdot(\boldsymbol{R}\,(-\tau) - \boldsymbol{r})\right]\right\}$$

$$\cdot\, \delta(\boldsymbol{P}(-\tau) - \boldsymbol{p}')\, f_a(\boldsymbol{p}, t)\,. \tag{49.28}$$

Hence

$$\left(\delta E\cdot\delta E\right)^{\text{source}}_{\omega,\boldsymbol{k}} = \sum_a \frac{(4\pi)^2 e_a^2 n_a}{k^2}$$

$$\cdot\, 2\,\mathrm{Re}\int_0^\infty d\tau \int d\boldsymbol{p}\, \exp\left\{-\Delta\tau + i\left[\omega\tau + \boldsymbol{k}\cdot(\boldsymbol{R}(-\tau) - \boldsymbol{r})\right]\right\} f_a(\boldsymbol{p}, t)\,. \tag{49.29}$$

We use again (49.22) and the expansion (49.24), integrate over ϕ and use (49.25). We note :

$$2\,\mathrm{Re}\int_0^\infty d\tau\,\exp\left[-\Delta\tau+i(\omega-k^{\parallel}v^{\parallel}-m\Omega_a)\,\tau\right]=2\pi\,\delta(\omega-k^{\parallel}v^{\parallel}-m\Omega_a)\tag{49.30}$$

and we find the desired expression :

$$\left(\delta\boldsymbol{E}\cdot\delta\boldsymbol{E}\right)^{\text{source}}_{\omega,\boldsymbol{k}}=\sum_a\frac{(4\pi)^2\,e_a^2n_a}{k^2}\sum_m\int dp^{\perp}\,2\pi p^{\perp}\int dp^{\parallel}$$
$$\cdot\,2\pi\,\delta(\omega-k^{\parallel}v^{\parallel}-m\Omega_a)\,I_m^2\left(\frac{k^{\perp}v^{\perp}}{\Omega_a}\right)f_a(p^{\parallel},p^{\perp},t)\,.\tag{49.31}$$

Substituting this expression into (49.27) we obtain the spectral density of the field fluctuations.

We now investigate the spectral density $\mathrm{Re}\,(\delta N_a\,\delta\boldsymbol{E})_{\omega,\boldsymbol{k},\boldsymbol{p}}$ which defines the collision integral. From (49.16) we obtain

$$\mathrm{Re}\left(\delta N_a\,\boldsymbol{k}\cdot\delta\boldsymbol{E}\right)_{\omega,\boldsymbol{k},\boldsymbol{p}}=\mathrm{Re}\left(\delta N_a\,\boldsymbol{k}\cdot\delta\boldsymbol{E}\right)^{\text{source}}_{\omega,\boldsymbol{k},\boldsymbol{p}}$$
$$-e_an_a\,\mathrm{Re}\int_0^\infty d\tau\,\exp\left\{-\Delta\tau+i\left[\omega\tau+\boldsymbol{k}\cdot\boldsymbol{R}(-\tau)-\boldsymbol{k}\cdot\boldsymbol{r}\right]\right\}$$
$$\cdot\,k_i\left(\delta E_j\,\delta E_i\right)_{\omega,\boldsymbol{k}}\;\frac{\partial f_a(\boldsymbol{P}(-\tau))}{\partial p_j}\,.\tag{49.32}$$

We first consider the second term. We can average it over the angle ϕ, as the collision integral does not depend on this variable. From (49.17) follows that $\delta\boldsymbol{E}$ is parallel to $\boldsymbol{k}$, thus

$$\left(\delta E_i\,\delta E_j\right)_{\omega,\boldsymbol{k}}=\frac{k_ik_j}{k^2}\left(\delta\boldsymbol{E}\cdot\delta\boldsymbol{E}\right)_{\omega,\boldsymbol{k}}\tag{49.33}$$

and this tensor can be determined from the spectral density of the field fluctuations.

We now transform the time integration in (49.32). We substitute (49.20), (49.22), (49.23), average over the angle, and perform the τ-integration, using Eqs (49.25). The second term thus becomes :

$$-\pi\,e_an_a\sum_m I_m^2\left(\frac{k^{\perp}v^{\perp}}{\Omega_a}\right)\delta(\omega-k^{\parallel}v^{\parallel}-m\Omega_a)\left(\delta\boldsymbol{E}\cdot\delta\boldsymbol{E}\right)_{\omega,\boldsymbol{k}}$$
$$\cdot\left(k^{\parallel}\frac{\partial}{\partial p^{\parallel}}+\frac{m\Omega_a}{v^{\perp}}\right)f_a(p^{\parallel},p^{\perp},t)\tag{49.34}$$

We now transform the first term on the right-hand side of (49.32). Using (49.18) we have :

$$\mathrm{Re}\left(\delta N_a\ \boldsymbol{k}\cdot\delta\boldsymbol{E}\right)^{\text{source}}_{\omega,\boldsymbol{k},\boldsymbol{p}} = \mathrm{Re}\,\frac{i}{\varepsilon^*(\omega,\boldsymbol{k})}\sum_b 4\pi e_b \int d\boldsymbol{p}'\left(\delta N_a\ \delta N_b\right)^{\text{source}}_{\omega,\boldsymbol{k},\boldsymbol{p},\boldsymbol{p}'}. \tag{49.35}$$

We substitute (49.28), sum over b and integrate over $\boldsymbol{p}'$. Noting from (49.6) that $f_a(\boldsymbol{p}) = f_a(\boldsymbol{P}(-\tau))$ we find :

$$\mathrm{Re}\left(\delta N_a\ \boldsymbol{k}\cdot\delta\boldsymbol{E}\right)_{\omega,\boldsymbol{k},\boldsymbol{p}} = \mathrm{Re}\,\frac{i\,e_a n_a}{\varepsilon^*(\omega,\boldsymbol{k})}\ 2\ \mathrm{Re}\int_0^\infty d\tau\ \exp\left\{-\Delta\tau + i\left[\omega\tau + \boldsymbol{k}\cdot\boldsymbol{R}(-\tau) - \boldsymbol{k}\cdot\boldsymbol{r}\right]\right\} f_a(\boldsymbol{p},t) \tag{49.36}$$

We substitute Eq. (49.22), expand in Bessel functions, average over the angle, use (49.25) and integrate over τ. Using Eq. (49.30) we find the following expression :

$$\frac{1}{2\pi}\int_0^{2\pi} d\phi\ \mathrm{Re}\left(\delta N_a\ \boldsymbol{k}\cdot\delta\boldsymbol{E}\right)_{\omega,\boldsymbol{k},\boldsymbol{p}} = \frac{8\pi^2\, Im\,\varepsilon(\omega,\boldsymbol{k})}{|\varepsilon(\omega,\boldsymbol{k})|^2}\ \delta(\omega - k^{\parallel}v^{\parallel} - m\Omega_a)\ f_a\left(p^{\parallel},p^{\perp},t\right). \tag{49.37}$$

From (49.26) we have the imaginary part of the dielectric constant

$$Im\,\varepsilon(\omega,\boldsymbol{k}) = -\sum_b \frac{4\pi^2 e_b^2 n_b}{k^2}\int_0^\infty dp^{\perp}\ 2\pi p^{\perp}\int_{-\infty}^{\infty} dp^{\parallel}\sum_m I_m^2\left(\frac{k^{\perp}v^{\perp}}{\Omega_b}\right) \cdot\delta\left(\omega - k^{\parallel}v^{\parallel} - m\Omega_b\right)\left[k^{\parallel}\frac{\partial}{\partial p^{\parallel}} + \frac{m\Omega_b}{v^{\perp}}\frac{\partial}{\partial p^{\perp}}\right] f_b\left(p^{\parallel},p^{\perp},t\right). \tag{49.38}$$

Equations (49.32), (49.34) (49.38) define the spectral density $(\delta N_a\ \delta\boldsymbol{E})_{\omega,\boldsymbol{k},\boldsymbol{p}}$, averaged over ϕ. We integrate this expression over ω, in order to obtain the spatial spectral density, with the result :

$$\frac{1}{2\pi}\int_0^{2\pi} d\phi\ \mathrm{Re}(\delta N_a\ \boldsymbol{k}\cdot\delta\boldsymbol{E})_{\boldsymbol{k},\boldsymbol{p}} = -\frac{e_a n_a}{2}\sum_m I_m^2\left(\frac{k^{\perp}v^{\perp}}{\Omega_a}\right) \cdot\left\{\left(\delta\boldsymbol{E}\cdot\delta\boldsymbol{E}\right)_{\omega,\boldsymbol{k}}\left[\boldsymbol{k}\cdot\frac{\partial}{\partial\boldsymbol{p}}\right] + \frac{8\pi\ \mathrm{Im}\,\varepsilon(\omega,\boldsymbol{k})}{|\varepsilon(\omega,\boldsymbol{k})|^2}\right\}_{\omega = k^{\parallel}v^{\parallel}+m\Omega_a} f_a\left(p^{\parallel},p^{\perp},t\right) \tag{49.39}$$

where we introduced the abbreviation

$$\left[\boldsymbol{k}\cdot\frac{\partial}{\partial\boldsymbol{p}}\right] \equiv k^{\parallel}\frac{\partial}{\partial p^{\parallel}} + \frac{m\Omega_a}{v^{\perp}}\frac{\partial}{\partial p^{\perp}}\ . \tag{49.40}$$

50. KINETIC EQUATION FOR A PLASMA IN THE PRESENCE OF AN EXTERNAL MAGNETIC FIELD

We write Eq. (49.7) for the collision integral in the form

$$I_a = -\frac{e_a}{n_a}\frac{1}{(2\pi)^3}\,\mathrm{Re}\int d\boldsymbol{k}\,\frac{\boldsymbol{k}}{k^2}\cdot\frac{\partial}{\partial \boldsymbol{p}}\,(\delta N_a\,\boldsymbol{k}\cdot\delta \boldsymbol{E})_{\boldsymbol{k},\,\boldsymbol{p}} \tag{50.1}$$

As the left-hand side of the equation does not depend on ϕ, we average (50.1) over this angle. In order to obtain an explicit result we proceed as follows. We use, as in section 49, cylindrical coordinates with the z-axis along $\boldsymbol{B}$: ϕ is the angle between $\boldsymbol{B}$ and $\boldsymbol{p}$. Then

$$\boldsymbol{k}\cdot\frac{\partial}{\partial \boldsymbol{p}} = k^{\parallel}\frac{\partial}{\partial p^{\parallel}} + \cos(\phi-\psi)\,k^{\perp}\frac{\partial}{\partial p^{\perp}} + \sin(\phi-\psi)\,\frac{k^{\perp}}{p^{\perp}}\frac{\partial}{\partial\phi} \tag{50.2}$$

We represent the spectral density appearing in (50.1) in the form

$$\mathrm{Re}\Big(\delta N_a\,\boldsymbol{k}\cdot\delta\boldsymbol{E}\Big)_{\boldsymbol{k},\boldsymbol{p}}$$

$$= \exp\Big[ix\sin(\phi-\psi)\Big]\sum_m \exp\Big[-im(\phi-\psi)\Big]\Big(\delta N_a\,\boldsymbol{k}\cdot\delta\boldsymbol{E}\Big)^{(m)}_{\boldsymbol{k},\boldsymbol{p}}$$

$$x \equiv \frac{k^{\perp}v^{\perp}}{\Omega_a}\,. \tag{50.3}$$

In order to find $\Big(\delta N_a\,\boldsymbol{k}\cdot\delta\boldsymbol{E}\Big)^{(m)}_{\boldsymbol{k},\boldsymbol{p}}$ we substitute (49.24) into (50.3) and average over ϕ:

$$\frac{1}{2\pi}\int_0^{2\pi} d\phi\;\mathrm{Re}\Big(\delta N_a\,\boldsymbol{k}\cdot\delta\boldsymbol{E}\Big)_{\boldsymbol{k},\boldsymbol{p}} = \sum_m I_m\Big(\frac{k^{\perp}v^{\perp}}{\Omega_a}\Big)\,\mathrm{Re}\Big(\delta N_a\,\boldsymbol{k}\cdot\delta\boldsymbol{E}\Big)^{(m)}_{\boldsymbol{k},\boldsymbol{p}} \tag{50.4}$$

Comparing Eqs (50.4), (49.39) we find

$$\mathrm{Re}\Big(\delta N_a\,\boldsymbol{k}\cdot\delta\boldsymbol{E}\Big)^{(m)}_{\boldsymbol{k},\boldsymbol{p}} = -\pi e_a n_a I_m\Big(\frac{k^{\perp}v^{\perp}}{\Omega_a}\Big)$$

$$\cdot\left\{\Big(\delta\boldsymbol{E}\cdot\delta\boldsymbol{E}\Big)_{\omega,\boldsymbol{k}}\Big[\boldsymbol{k}\cdot\frac{\partial}{\partial\boldsymbol{p}}\Big] + \frac{8\pi\,\mathrm{Im}\,\varepsilon(\omega,\boldsymbol{k})}{|\varepsilon(\omega,\boldsymbol{k})|^2}\right\}_{\omega=k^{\parallel}v^{\parallel}+m\Omega_a} f_a(p^{\parallel},p^{\perp},t)\,. \tag{50.5}$$

We now substitute (50.2), (£0.3) into (50.1) and average over ϕ. Using (49.24) and the results

$$\sum_n I_n(x)\,\frac{1}{2\pi}\int_0^{2\pi} d\phi\,\exp\Big[in(\phi-\psi) - im(\phi-\psi)\Big] = I_m(x) \tag{50.6}$$

$$\sum_n I_n(x)\,\frac{1}{2\pi}\int_0^{2\pi} d\phi\,\cos(\phi-\psi)\exp\Big[in(\phi-\psi) - im(\phi-\psi)\Big]$$

$$= \frac{1}{2}\Big[I_{m+1}(x) + I_{m-1}(x)\Big] = \frac{m}{x}\,I_m(x)\,, \tag{50.7}$$

we obtain

$$\frac{1}{2\pi}\int_0^{2\pi} d\phi\, I_a(\boldsymbol{p},t) = -\frac{e_a}{n_a}\frac{1}{(2\pi)^3}\sum_m \int d\boldsymbol{k}\left[\boldsymbol{k}\cdot\frac{\partial}{\partial \boldsymbol{p}}\right] I_m\left(\frac{k^\perp v^\perp}{\Omega_a}\right)\left(\delta N_a\, \boldsymbol{k}\cdot\delta\boldsymbol{E}\right)^{(m)}_{\boldsymbol{k},\boldsymbol{p}} . \quad (50.8)$$

Substituting Eq. (50.5) we find the desired expression for the collision integral. We write it in two forms. The first one is the Fokker-Planck operator :

$$I_a(p^\parallel, p^\perp, t) = \sum_m \int d\boldsymbol{k}\left\{\left[\boldsymbol{k}\cdot\frac{\partial}{\partial\boldsymbol{p}}\right] D_a^{(m)}(p^\parallel, p^\perp, t)\left[\boldsymbol{k}\cdot\frac{\partial}{\partial\boldsymbol{p}}\right] f_a(p^\parallel, p^\perp, t)\right.$$

$$\left. + \left[\boldsymbol{k}\cdot\frac{\partial}{\partial\boldsymbol{p}}\right] A_a^{(m)}(p^\parallel, p^\perp, t)\, f_a(p^\parallel, p^\perp, t)\right\} . \quad (50.9)$$

The coefficients are defined as follows

$$D_a^{(m)} = \frac{e_a^2}{16\pi^3 k^2} I_m^2\left(\frac{k^\perp v^\perp}{\Omega_a}\right)\left(\delta\boldsymbol{E}\cdot\delta\boldsymbol{E}\right)_{k^\parallel v^\parallel + m\Omega_a, \boldsymbol{k}} \quad (50.10)$$

$$A_a^{(m)} = \frac{e_a^2}{2\pi^2 k^2} I_m^2\left(\frac{k^\perp v^\perp}{\Omega_a}\right)\frac{\operatorname{Im}\varepsilon(k^\parallel v^\parallel + m\Omega_a, \boldsymbol{k})}{|\varepsilon(k^\parallel v^\parallel + m\Omega_a, \boldsymbol{k})|^2} . \quad (50.11)$$

As in section 37, the diffusion coefficient is defined in terms of the spectral density of the field fluctuations, whereas the friction coefficient is proportional to the imaginary part of the dielectric constant.

In order to obtain another form of the collision integral, analogous to (37.1), we use Eqs (49.27) and (49.38) and find

$$I_a(p^\parallel, p^\perp, t) = \sum_b n_b \sum_m \sum_{m'} \int dp'^\perp\, 2\pi p'^\perp \int dp'^\parallel \int dk^\perp\, 2\pi k^\perp \int dk^\parallel$$

$$\cdot \left[\boldsymbol{k}\cdot\frac{\partial}{\partial\boldsymbol{p}}\right]_m Q^{ab}_{mm'}\left\{\left[\boldsymbol{k}\cdot\frac{\partial}{\partial\boldsymbol{p}}\right]_m\right.$$

$$\left. - \left[\boldsymbol{k}\cdot\frac{\partial}{\partial\boldsymbol{p}'}\right]_{m'}\right\} f_a(p^\parallel, p^\perp, t)\, f_b(p'^\parallel, p'^\perp, t) \quad (50.12)$$

where

$$Q^{ab}_{mm'} = 2e_a^2 e_b^2 I_m^2\left(\frac{k^\perp v^\perp}{\Omega_a}\right) I_{m'}^2\left(\frac{k^\perp v'^\perp}{\Omega_b}\right)$$

$$\cdot \frac{\delta(k^\parallel v^\parallel + m\Omega_a - k^\parallel v'^\parallel - m'\Omega_b)}{k^4 |\varepsilon(k^\parallel v^\parallel + m\Omega_a, \boldsymbol{k})|^2} . \quad (50.13)$$

The kinetic equation for a plasma in a magnetic field, taking the polarization processes into account was considered in the work of Rostoker [36], Eleonsky, Zyryanov and Silin [37], and in references [38.8]. In earlier work, Belyaev [39] studied the kinetic equation of a plasma in a magnetic field without taking polarization into account, i.e., an approximation analogous to the Landau approximation.

In the Landau approximation there is a divergence in the $\boldsymbol{k}$-integration in the collision integral. Therefore we must put the limits (38.2) on the integration range. The lower limit corresponds to an approximate account of the polarization.

In the presence of a strong magnetic field, when the Larmor frequency and radius, Ω_L, r_L are such that

$$\Omega_L > \omega_L \quad , \qquad R_L < r_{\mathrm{D}} \tag{50.14}$$

the inclusion of polarization effects is not really necessary, as in that case the divergence at small k disappears; therefore the effective value $k_{\mathbf{min}}$ is $\sim R_L^{-1} < r_{\mathrm{D}}^{-1}$.

Thus, the influence of the polarization in the collision integral is important when

$$\Omega_L < \omega_L \quad , \qquad R_L > r_{\mathrm{D}} \ . \tag{50.15}$$

The expressions (50.9), (50.12) for the collision integral are very complicated. They can be significantly simplified if the effective potential (sect. 47) is used, thus taking into account the averaged contribution of the dynamical polarization.

We note that in the Landau approximation the limitation (49.6) on the form of f_a is not necessary; one can also take into account the fast variation of the distribution function. There is an undoubted interest in deriving the kinetic equation for a plasma in a magnetic field, taking polarization into account, and without the limitation (49.6).

The methods used in this chapter can also be applied to quantum systems. In that case, unfortunately, we cannot give the corresponding results. We only note that a substantial contribution to the investigation of processes in strong magnetic fields was made by Pavel Stepanovich Zyryanov, who died tragically in the winter of 1974.

CHAPTER 9

The Spatially Homogeneous Nonideal Plasma

51. SPECTRAL DENSITIES OF NON-STATIONARY PROCESSES

For the calculation of the spectral densities defining the collision integrals of nonideal plasmas it is necessary to take account of the retardation of the distribution functions. In other words, one may not neglect the change of the distribution functions over a correlation time of the fluctuations defining the kinetic processes. As a result, the fluctuation process becomes non-stationary, and we must devise methods for the calculation of spectral densities of such processes. We first consider some examples.

Consider a process characterized by the following stochastic functions of time:

$$x_1(t)\ ,\ x_2(t)\ ,\ldots,\ x_n(t)\ . \tag{51.1}$$

We assume that the average of these functions vanishes, and that their (one-time) second moments are given functions of time. For a stationary process, the latter would be time-independent. We introduce now the following two-time correlation functions:

$$\begin{aligned} \phi_{ij}(t,t-\tau) &\equiv \phi_{ij}(t,\tau) = \langle x_i(t)\ x_j(t-\tau)\rangle \\ \phi_{ji}(t,\ t-\tau) &\equiv \phi_{ji}(t,\tau) = \langle x_j(t)\ x_i(t-\tau)\rangle\ . \end{aligned} \tag{51.2}$$

For a stationary process, these functions ϕ_{ij} do not depend on t, and obey the following relations [45]

$$\langle x_i(t)\ x_j(t-\tau)\rangle = \langle x_i(t+\tau)\ x_j(t)\rangle\ , \tag{51.3}$$

i.e., the correlations are invariant under the translation: $t \to t+\tau$. We then conclude that, for a stationary process,

$$\phi_{ij}(\tau) = \phi_{ji}(-\tau)\ . \tag{51.4}$$

In the linear approximation, the functions $\phi_{ij}(t, t-\tau)$ obey the following equations:

$$\begin{aligned} &\frac{d}{dt}\phi_{ij}(t, t-\tau) + \lambda_{ik}\phi_{kj}(t, t-\tau) = 0, \quad \tau > 0 \\ &\frac{d}{dt}\phi_{ji}(t, t-\tau) + \lambda_{jk}\phi_{ki}(t, t-\tau) = 0, \quad \tau > 0 \end{aligned} \tag{51.5}$$

which must be solved with the initial conditions :

$$\begin{aligned} &\phi_{ij}(t, t-\tau)\Big|_{t=t-\tau} = \langle x_i x_j \rangle_{t-\tau} \\ &\phi_{ji}(t, t-\tau)\Big|_{t=t-\tau} = \langle x_j x_i \rangle_{t-\tau}. \end{aligned} \tag{51.6}$$

For a non-stationary process, the spectral density is defined as follows:

$$(x_i x_j)_{\omega, t} = \int_0^\infty d\tau \left\{ \langle x_i x_j \rangle_{t, t-\tau} e^{i\omega\tau} + \langle x_j x_i \rangle_{t, t-\tau} e^{-i\omega\tau} \right\}. \tag{51.7}$$

It follows that

$$(x_i x_j)_{\omega, t} = \left[(x_j x_i)_{\omega, t} \right]^*. \tag{51.8}$$

Thus the spectral densities of a non-stationary process are defined as one-sided Fourier transforms in τ of the correlations $\phi_{ij}(t, t-\tau)$.

In the case of a stationary process, we may use (51.3) in (51.7) and change τ into $-\tau$ in the second integral: we then obtain the usual definition :

$$(x_i x_j)_\omega = \int_{-\infty}^{\infty} d\tau \langle x_i x_j \rangle_{t, t-\tau} e^{i\omega\tau} \equiv \int_{-\infty}^{\infty} d\tau\, \phi_{ij}(\tau)\, e^{i\omega\tau}. \tag{51.9}$$

The Fourier transforms of the correlation functions are

$$\begin{aligned} &\langle x_i x_j \rangle_{t, t-\tau} = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega\, (x_i x_j)_{\omega, t}\, e^{-i\omega\tau} \\ &\langle x_j x_i \rangle_{t, t-\tau} = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega (x_j x_i)^*_{\omega, t}\, e^{i\omega\tau}. \end{aligned} \tag{51.10}$$

From the condition of reality of the correlations, we have

$$\begin{aligned} &(x_i x_j)^*_{\omega, t} = (x_i x_j)_{-\omega, t} \\ &(x_j x_i)^*_{\omega, t} = (x_j x_i)_{-\omega, t}. \end{aligned} \tag{51.11}$$

We now consider as an example the case of one random variable. In this case, instead of the two functions (51.2) we only have one :

$$\phi(t, t-\tau) \equiv \phi(t, \tau) = \langle x x \rangle_{t, t-\tau}. \tag{51.12}$$

It obeys the equation

$$\frac{d}{dt}\phi(t, t-\tau) + \lambda\phi(t, t-\tau) = 0, \quad \phi(t, t-\tau)\Big|_{t=t-\tau} = \langle x^2 \rangle_{t-\tau}. \tag{51.13}$$

Its solution is

$$\phi(t,t-\tau) = e^{-\lambda\tau}\langle x^2\rangle_{t-\tau}\ , \qquad \tau \geqslant 0 \tag{51.14}$$

and from (51.7) :

$$(x^2)_{\omega,t} = 2\ \mathrm{Re}\int_0^\infty d\tau\ (x\,x)_{t,t-\tau}\ e^{i\omega\tau} \tag{51.15}$$

Substituting (51.14) into (51.15) we obtain the spectral density of the non-stationary process:

$$(x^2)_{\omega,t} = 2\ \mathrm{Re}\int_0^\infty d\tau\ e^{-i\lambda\tau+i\omega\tau}\ \langle x^2\rangle_{t-\tau}\ . \tag{51.16}$$

For a stationary process, this expression reduces to

$$(x^2)_\omega = \frac{2\lambda\,\langle x^2\rangle}{\omega^2+\lambda^2}\ . \tag{51.17}$$

The expression (51.16) for the spectral density of the non-stationary process can be represented in the form:

$$(x^2)_{\omega,t} = \int_{-\infty}^{\infty} d\tau\, e^{-\lambda|\tau|}\ e^{i\omega\tau}\ \langle x^2\rangle_{t-|\tau|}\ . \tag{51.18}$$

Both expressions (51.16) and (51.18) satisfy the condition :

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega\,(x^2)_{\omega,t} = \langle x^2\rangle_t\ . \tag{51.19}$$

We note that the general expression (51.15) can also be written in a form analogous to (51.18) :

$$(x^2)_{\omega,t} = \int_{-\infty}^{\infty} d\tau\,\langle x\,x\rangle_{t,t-|\tau|}\ e^{i\omega\tau} \equiv \int_{-\infty}^{\infty} d\tau\,\langle x\,x\rangle_{t,\ |\tau|}\ e^{i\omega\tau}\ . \tag{51.20}$$

The corresponding inverse transformation is

$$\langle x\,x\rangle_{t,t-|\tau|} = (2\pi)^{-1}\int_{-\infty}^{\infty} d\omega\ (x^2)_{\omega,t}\ e^{-i\omega|\tau|}\ . \tag{51.21}$$

We now consider the corresponding Langevin equation. For the case of one variable it is :

$$\frac{dx}{dt} + \lambda x = \xi(t)\ , \tag{51.22}$$

where ξ is a random source :

$$\langle\xi\rangle = 0\ , \qquad \langle\xi\,\xi\rangle_{t,t'} = B(t)\,\delta(t-t')\ . \tag{51.23}$$

We now derive the relation between the intensity of the source fluctuations, $B(t)$, and $\langle x^2\rangle_t$ for a non-stationary process. We use the solution of the Langevin equation

$$x(t) = \int_{-\infty}^{t} dt'\, e^{-\lambda(t-t')}\ \xi(t') \tag{51.24}$$

and from (51.23) we obtain :

$$\langle x^2 \rangle_t = \int_0^\infty d\tau \, e^{-2\lambda\tau} B(t-\tau) \, . \tag{51.25}$$

Thus, in the non-stationary case, the relation between $\langle x^2 \rangle_t$ and $B(t)$ exhibits the retardation effect.

From (51.25) we may derive the differential relation

$$\frac{d}{dt} \langle x^2 \rangle_t + 2\lambda \langle x^2 \rangle_t = B(t) \, . \tag{51.26}$$

In the stationary case, both (51.25) and (51.26) lead to the well-known result:

$$B = 2\lambda \langle x^2 \rangle \, . \tag{51.27}$$

Corresponding relations between the spectral densities can also be derived. We substitute Eq. (51.25) into (51.16) and perform an integration by parts. As a result we find an equation for the spectral density

$$\frac{d}{dt} (x^2)_{\omega,t} + 2\lambda \, (x^2)_{\omega,t} = 2 \,\mathrm{Re} \int_0^\infty d\tau \, e^{-\lambda\tau + i\omega\tau} B(t-\tau) . \tag{51.28}$$

If we integrate over ω and use (51.19), we recover (51.26).

For a stationary process, Eq. (51.28) yields the relation [45] :

$$(x^2)_\omega = \frac{B}{\omega^2 + \lambda^2} \, . \tag{51.29}$$

Because of (51.27) this coincides with (51.17). From the definition (51.23) follows that, for a stationary process :

$$\langle \xi\xi \rangle_\tau = B\delta(\tau) \, , \quad (\xi\xi)_\omega = B \, , \tag{51.30}$$

hence the relation (51.29) can also be written in the form

$$(x^2)_\omega = \frac{(\xi^2)_\omega}{\omega^2 + \lambda^2} \, . \tag{51.31}$$

This equation provides the link between the spectral densities $(x^2)_\omega$ and $(\xi^2)_\omega$ for a stationary process.

For a non-stationary process, the spectral density of the source is defined as follows :

$$2\lambda (x^2)^{\mathbf{source}}_{\omega,t} = 2 \,\mathrm{Re} \int_0^\infty d\tau \, e^{-\lambda\tau + i\omega\tau} B(t-\tau) \, . \tag{51.32}$$

Equation (51.28) can thus be written in the form

$$\frac{d(x^2)_{\omega,t}}{dt} = -2\lambda \left[(x^2)_{\omega,t} - (x^2)^{\mathbf{source}}_{\omega,t} \right] . \tag{51.33}$$

We may obtain still another convenient form of the equations for the spectral density. From the Langevin equation (51.22) we find :

$$\left\langle \frac{dx}{dt}\, x \right\rangle_{t,\,t-\tau} = -\lambda \left\langle xx \right\rangle_{t,\,t-\tau} + \left\langle \xi\, x \right\rangle_{t,\,t-\tau}$$
$$\left\langle x\, \frac{dx}{dt} \right\rangle_{t,\,t-\tau} = -\lambda \left\langle xx \right\rangle_{t,\,t-\tau} + \left\langle x\,\xi \right\rangle_{t,\,t-\tau} . \qquad (51.34)$$

Combining these equations with (51.7) we obtain

$$\left(\frac{dx}{dt}\, x \right)_{\omega,\,t} = -\lambda\, (x^2)_{\omega,\,t} + (\xi x)_{\omega,\,t}$$
$$\left(x\, \frac{dx}{dt} \right)_{\omega,\,t} = -\lambda\, (x^2)_{\omega,\,t} + (x\xi)_{\omega,\,t} . \qquad (51.35)$$

Adding these equations, we obtain

$$\frac{d}{dt}\,(x^2)_{\omega,\,t} + 2\lambda\,(x^2)_{\omega,\,t} = (\xi\, x)_{\omega\,,\,t} + (x\xi)_{\omega,\,t} = 2\ \mathrm{Re}\,(x\xi)_{\omega,\,t} . \qquad (51.36)$$

From the general formula (51.7) we get :

$$(x\xi)_{\omega,\,t} = \int_0^\infty d\tau \left[\left\langle x\xi \right\rangle_{t,\,t-\tau} e^{i\omega\tau} + \left\langle \xi\, x \right\rangle_{t,\,t-\tau} e^{-i\omega\tau} \right] . \qquad (51.37)$$

Thus, in order to define the spectral density, we need the two functions

$$\left\langle x\xi \right\rangle_{t,\,t-\tau} , \quad \left\langle \xi\, x \right\rangle_{t,\,t-\tau} .$$

Using the Langevin equation (51.24) and (51.23) we find

$$\left\langle x\xi \right\rangle^{\bullet}_{t,\,t-\tau} = \int_0^\infty d\tau'\, e^{-\lambda\tau'} \left\langle \xi\xi \right\rangle_{t-\tau',\,t-\tau} = e^{-\lambda\tau}\, B(t-\tau) \qquad (51.38)$$

$$\left\langle \xi x \right\rangle_{t,\,t-\tau} = \int_0^\infty d\tau'\, e^{-\lambda\tau'} \left\langle \xi\xi \right\rangle_{t,\,t-\tau-\tau'} = \int_0^\infty d\tau'\, e^{-\lambda\tau'}\, B(t)\, \delta(\tau+\tau') = 0 . \qquad (51.39)$$

Collecting the results (51.37) – (51.39) we get :

$$2\ \mathrm{Re}\,(x\xi)_{\omega,\,t} = 2\,\mathrm{Re} \int_0^\infty d\tau\, e^{-\lambda\tau + i\omega\tau}\, B(t-\tau) . \qquad (51.40)$$

Substituting this expression into the right-hand side of (51.36) we recover Eq. (51.28).

52. SPECTRAL DENSITIES OF THE SOURCE FLUCTUATIONS IN A NONIDEAL PLASMA

Corresponding to (51.7), the spectral density of the fluctuations $\delta N_a, \delta N_b$ for a nonideal. spatially inhomogeneous plasma is expressed as follows, in terms of the correlations :

$$\left(\delta N_a\, \delta N_b \right)_{\omega,\,\boldsymbol{k},\,t,\,\boldsymbol{r},\,\boldsymbol{p},\,\boldsymbol{p}'} = \int_0^\infty d(\boldsymbol{r}-\boldsymbol{r}')$$
$$\cdot \left\{ \left\langle \delta N_a\, \delta N_b \right\rangle_{t,\,t-\tau,\,\boldsymbol{r},\,\boldsymbol{r}-\boldsymbol{r}',\,\boldsymbol{p},\,\boldsymbol{p}'} e^{i(\omega\tau - \boldsymbol{k}\cdot\boldsymbol{r} + \boldsymbol{k}\cdot\boldsymbol{r}')} \right.$$
$$\left. + \left\langle \delta N_b\, \delta N_a \right\rangle_{t,\,t-\tau,\,\boldsymbol{r},\,\boldsymbol{r}-\boldsymbol{r}',\,\boldsymbol{p},\,\boldsymbol{p}'} e^{-i(\omega\tau - \boldsymbol{k}\cdot\boldsymbol{r} + \boldsymbol{k}\cdot\boldsymbol{r}')} \right\} . \qquad (52.1)$$

We thus have the following symmetry property :

$$\left(\delta N_a\ \delta N_b\right)_{\omega,\boldsymbol{k},t,\boldsymbol{r},\boldsymbol{p},\boldsymbol{p}'} = \left[\left(\delta N_b\ \delta N_a\right)_{\omega,\boldsymbol{k},t,\boldsymbol{r},\boldsymbol{p}',\boldsymbol{p}}\right]^* \tag{52.2}$$

We obtain a convenient form of (52.1) by introducing the symbol :

$$\left(\delta N_a\ \delta N_b\right)^+_{\omega,\boldsymbol{k},t,\boldsymbol{r},\boldsymbol{p},\boldsymbol{p}'} = \int_0^\infty d\tau \int d(\boldsymbol{r}-\boldsymbol{r}')\left\langle\delta N_a\ \delta N_b\right\rangle_{t,t-\tau,\boldsymbol{r},\boldsymbol{r}-\boldsymbol{r}',\boldsymbol{p},\boldsymbol{p}'} e^{i(\omega\tau-\boldsymbol{k}\cdot\boldsymbol{r}+\boldsymbol{k}\cdot\boldsymbol{r}')} \tag{52.3}$$

It then follows that

$$\left(\delta N_a\ \delta N_b\right)_{\omega,\boldsymbol{k},t,\boldsymbol{r},\boldsymbol{p},\boldsymbol{p}'} = \left(\delta N_a\ \delta N_b\right)^+_{\omega,\boldsymbol{k},t,\boldsymbol{r},\boldsymbol{p},\boldsymbol{p}'} + \left[\left(\delta N_b\ \delta N_a\right)_{\omega,\boldsymbol{k},t,\boldsymbol{r},\boldsymbol{p}',\boldsymbol{p}}\right]^* . \tag{52.4}$$

For a stationary and homogeneous process we have the relation :

$$\left\langle\delta N_a\ \delta N_b\right\rangle_{t,t-\tau,\boldsymbol{r},\boldsymbol{r}-\boldsymbol{r}',\boldsymbol{p},\boldsymbol{p}'} = \left\langle\delta N_a\ \delta N_b\right\rangle_{t+\tau,t,\boldsymbol{r}+\boldsymbol{r}',\boldsymbol{r},\boldsymbol{p},\boldsymbol{p}'} \tag{52.5}$$

analogous to (51.3). Equation (52.1) reduces in this case to :

$$\begin{aligned}\left(\delta N_a\delta N_b\right)_{\omega,\boldsymbol{k},\boldsymbol{p},\boldsymbol{p}'} &= \int_{-\infty}^{\infty} d\tau \int d(\boldsymbol{r}-\boldsymbol{r}')\left\langle\delta N_a\delta N_b\right\rangle_{t,t-\tau,\boldsymbol{r},\boldsymbol{r}-\boldsymbol{r}',\boldsymbol{p},\boldsymbol{p}'} e^{i(\omega\tau-\boldsymbol{k}\cdot\boldsymbol{r}+\boldsymbol{k}\cdot\boldsymbol{r}')} \\ &= \int_{-\infty}^{\infty} d\tau \left(\delta N_a\ \delta N_b\right)_{t,t-\tau,\boldsymbol{k},\boldsymbol{p},\boldsymbol{p}'} e^{i\omega\tau}\end{aligned} \tag{52.6}$$

In this chapter we study the kinetic equation for a nonideal, but spatially homogeneous plasma. The corresponding random process is homogeneous, but non-stationary. The spectral density is then defined by

$$\begin{aligned}\left(\delta N_a\ \delta N_b\right)_{\omega,\boldsymbol{k},t,\boldsymbol{p},\boldsymbol{p}'} &= \int_0^\infty d\tau \left[\left(\delta N_a\ \delta N_b\right)_{t,t-\tau,\boldsymbol{k},\boldsymbol{p},\boldsymbol{p}'} e^{i\omega\tau} + \left(\delta N_b\ \delta N_a\right)_{t,t-\tau,-\boldsymbol{k},\boldsymbol{p}',\boldsymbol{p}} e^{-i\omega\tau}\right] \\ &\equiv \left(\delta N_a\ \delta N_b\right)^+_{\omega,\boldsymbol{k},t,\boldsymbol{p},\boldsymbol{p}'} + \left[\left(\delta N_b\ \delta N_a\right)^+_{\omega,\boldsymbol{k},t,\boldsymbol{p}',\boldsymbol{p}}\right]^* .\end{aligned} \tag{52.7}$$

This equation will be used now for the derivation of the spectral density of the

source fluctuations. From (34.1) together with the initial condition (33.11) follows the equation for the spatial spectral density of the fluctuations $\delta N_a^{\text{source}}$:

$$\left(\frac{\partial}{\partial t} + i\boldsymbol{k}\cdot\boldsymbol{v} + \Delta\right)\left(\delta N_a\, \delta N_b\right)^{\text{source}}_{t,t-\tau,\boldsymbol{k},\boldsymbol{p},\boldsymbol{p}'} = 0 \qquad (\tau > 0)$$

$$\left(\delta N_a\, \delta N_b\right)^{\text{source}}_{t,t-\tau,\boldsymbol{k},\boldsymbol{p},\boldsymbol{p}'}\Big|_{t=t-\tau} = n_a\, \delta_{ab}\, \delta(\boldsymbol{p}-\boldsymbol{p}')\, f_a(\boldsymbol{p}, t-\tau)\,. \tag{52.8}$$

The solution of this equation is :

$$\left(\delta N_a\, \delta N_b\right)^{\text{source}}_{t,t-\tau,\boldsymbol{k},\boldsymbol{p},\boldsymbol{p}'} = n_a\, \delta_{ab}\, \delta(\boldsymbol{p}-\boldsymbol{p}')\, e^{-\Delta\tau - i\boldsymbol{k}\cdot\boldsymbol{v}\tau}\, f_a(\boldsymbol{p}, t-\tau). \tag{52.9}$$

In the same way, we obtain an expression for the following function, appearing in (52.7) :

$$\left(\delta N_b\, \delta N_a\right)^{\text{source}}_{t,t-\tau,-\boldsymbol{k},\boldsymbol{p},\boldsymbol{p}'} = n_a\, \delta_{ab}\, \delta(\boldsymbol{p}-\boldsymbol{p}')\, e^{-\Delta\tau + i\boldsymbol{k}\cdot\boldsymbol{v}\tau}\, f_a(\boldsymbol{p}, t-\tau). \tag{52.10}$$

We then find from (52.7) the expression for the spectral density : [7]

$$\left(\delta N_a\, \delta N_b\right)^{\text{source}}_{\omega,\boldsymbol{k},t,\boldsymbol{p},\boldsymbol{p}'} = n_a\, \delta_{ab}\, \delta(\boldsymbol{p}-\boldsymbol{p}')\, 2\,\mathrm{Re}\int_0^\infty d\tau\, e^{-\Delta\tau + i(\omega-\boldsymbol{k}\cdot\boldsymbol{v})\tau}\, f_a(\boldsymbol{p}, t-\tau) \tag{52.11}$$

From this expression, to zeroth order in the retardation, we recover Eq.(34.11) for an ideal plasma.

From (52.11), we may derive the simpler spectral densities, such as: [7]

$$\left(\delta N_a\, \delta \boldsymbol{E}\right)^{\text{source}}_{\omega,\boldsymbol{k},t,\boldsymbol{p}} = \frac{i\boldsymbol{k}}{k^2}\, 4\pi e_a n_a\, 2\,\mathrm{Re}\int_0^\infty d\tau\, e^{-\Delta\tau + i(\omega-\boldsymbol{k}\cdot\boldsymbol{v})\tau}\, f_a(\boldsymbol{p}, t-\tau)$$

$$\left(\delta \boldsymbol{j}\cdot\delta \boldsymbol{E}\right)^{\text{source}}_{\omega,\boldsymbol{k},t} = 4\pi i \sum_a \frac{e_a^2 n_a}{k^2}\, 2\,\mathrm{Re}\int_0^\infty d\tau \int d\boldsymbol{p}\, e^{-\Delta\tau + i(\omega-\boldsymbol{k}\cdot\boldsymbol{v})\tau}\, \boldsymbol{k}\cdot\boldsymbol{v}\, f_a(\boldsymbol{p}, t-\tau)$$

$$\left(\delta \boldsymbol{E}\cdot\delta \boldsymbol{E}\right)^{\text{source}}_{\omega,\boldsymbol{k},t} = \sum_a \frac{(4\pi)^2 e_a^2 n_a}{k^2}\, 2\,\mathrm{Re}\int_0^\infty d\tau \int d\boldsymbol{p}\, e^{-\Delta\tau + i(\omega-\boldsymbol{k}\cdot\boldsymbol{v})\tau}\, f_a(\boldsymbol{p}, t-\tau). \tag{52.12}$$

Integrating Eqs (52.11), (52.12) over ω we find the corresponding spatial spectral densities. For instance,

$$\left(\delta N_a\, \delta N_b\right)^{\text{source}}_{\boldsymbol{k},t,\boldsymbol{p},\boldsymbol{p}'} = n_a\, \delta_{ab}\, \delta(\boldsymbol{p}-\boldsymbol{p}')\, f_a(\boldsymbol{p}, t) \tag{52.13}$$

In performing the ω-integration we used the identity :

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega \int_0^\infty d\tau\, e^{i\omega\tau} f(t-\tau) = \tfrac{1}{2} f(t)\,. \tag{52.14}$$

We now consider, instead of (52.8), the corresponding Langevin equation for $\delta N_a^{\text{source}}$:

$$\left(\frac{\partial}{\partial t} + i\boldsymbol{k}\cdot\boldsymbol{v} + \Delta\right) \delta N_a^{\mathbf{source}} = y_a(\boldsymbol{p}, \boldsymbol{k}, t) \tag{52.12}$$

Here y_a is a delta-correlated source. Its average is zero, and:

$$\langle y_a y_b \rangle_{t', t'', \boldsymbol{k}, \boldsymbol{p}, \boldsymbol{p}'} = B_{ab}(\boldsymbol{p}, \boldsymbol{p}', t')\ \delta(t' - t'') \tag{52.16}$$

where B_{ab} is the intensity of the fluctuations of y_a, y_b.

We now derive the relation between the intensity and the function (52.13). We use the solution of (52.15) :

$$\delta N_a^{\mathbf{source}}(\boldsymbol{k}, \boldsymbol{p}, t) = \int_{-\infty}^{t} dt'\, e^{-\Delta(t-t') - i\boldsymbol{k}\cdot\boldsymbol{v}(t-t')}\, y_a(\boldsymbol{k}, \boldsymbol{p}, t') \tag{52.17}$$

This expression, and the corresponding one for δN_b is combined with (52.16) to yield :

$$\left(\delta N_a\, \delta N_b\right)^{\mathbf{source}}_{\boldsymbol{k}, \boldsymbol{p}, \boldsymbol{p}', t} = \int_0^{\infty} d\tau\, e^{-2\Delta\tau - i(\boldsymbol{k}\cdot\boldsymbol{v} - \boldsymbol{k}\cdot\boldsymbol{v}')\tau}\, B_{ab}(\boldsymbol{p}, \boldsymbol{p}', t-\tau). \tag{52.18}$$

Comparing this equation with (52.13) we see that the function B_{ab} has the structure

$$B_{ab}(\boldsymbol{p}, \boldsymbol{p}', t) = \delta_{ab}\ \delta(\boldsymbol{p} - \boldsymbol{p}')\ n_a B_a(\boldsymbol{p}, t)\ . \tag{52.19}$$

Thus (52.18) reduces to

$$\left(\delta N_a\, \delta N_b\right)^{\mathbf{source}}_{\boldsymbol{k}, \boldsymbol{p}, \boldsymbol{p}', t} = \int_0^{\infty} d\tau\, e^{-2\Delta\tau}\, B_{ab}(\boldsymbol{p}, \boldsymbol{p}', t-\tau).$$

From this expression follows the differential relation

$$\left(\frac{d}{dt} + 2\Delta\right)\left(\delta N_a\ \delta N_b\right)^{\mathbf{source}}_{\boldsymbol{k}, t, \boldsymbol{p}, \boldsymbol{p}'} = B_{ab}(\boldsymbol{p}, \boldsymbol{p}', t)\ . \tag{52.20}$$

Using the relations (52.13), (52.19) we obtain the relation between B_a and f_a:

$$[(d/dt) + 2\Delta]\ f_a(\boldsymbol{p}, t) = B_a(\boldsymbol{p}, t)\ . \tag{52.21}$$

In the stationary state, we have

$$B_a(\boldsymbol{p}) = 2\Delta f_a(\boldsymbol{p}) \tag{52.22}$$

and thus

$$B_{ab}(\boldsymbol{p}, \boldsymbol{p}', t) = 2\Delta\left(\delta N_a\ \delta N_b\right)^{\mathbf{source}}_{\boldsymbol{k}, \boldsymbol{p}, \boldsymbol{p}', t} \tag{52.23}$$

Finally, we derive the relation between the function B_{ab} and the space-time spectral density (52.11). We eliminate the function f_a from the latter equation by means of (52.21). As a result we find

$$\left(\frac{d}{dt} + 2\Delta\right)\left(\delta N_a\ \delta N_b\right)^{\mathbf{source}}_{\omega, \boldsymbol{k}, t, \boldsymbol{p}, \boldsymbol{p}'}$$

$$= 2\,\mathrm{Re}\int_0^{\infty} d\tau\, e^{-\Delta\tau + i(\omega - \boldsymbol{k}\cdot\boldsymbol{v})\tau}\, B_{ab}(\boldsymbol{p}, \boldsymbol{p}', t-\tau). \tag{52.24}$$

Note that (42.21) and (52.24) correspond to Eqs (51.26) and (51.28).

Using (52.21) and (52.24) we can obtain corresponding relations for the spectral densities of the fluctuations of δN_a, $\delta \boldsymbol{E}$ and $\delta \boldsymbol{j}$. For instance, the equations for the spectral density of the field fluctuations are :

$$\left(\frac{\partial}{\partial t}+2\Delta\right)\left(\delta \boldsymbol{E}\cdot\delta \boldsymbol{E}\right)^{\text{source}}_{\boldsymbol{k},t} = \sum_a \frac{(4\pi)^2 e_a^2 n_a}{k^2}\int d\boldsymbol{p}\; B_a(\boldsymbol{p},t) \tag{52.25}$$

$$\left(\frac{\partial}{\partial t}+2\Delta\right)\left(\delta \boldsymbol{E}\cdot\delta \boldsymbol{E}\right)^{\text{source}}_{\omega,\boldsymbol{k},t}$$
$$=\sum_a \frac{(4\pi)^2 e_a^2 n_a}{k^2}\int_0^\infty d\tau\int d\boldsymbol{p}\; e^{-\Delta\tau+i(\omega-\boldsymbol{k}\cdot\boldsymbol{v})\tau}\, B_a(\boldsymbol{p},t-\tau)\,. \tag{52.26}$$

In the stationary case, Eq. (52.26), combined with (52.22) yields (34.14).

53. THE LANDAU KINETIC EQUATION FOR A NONIDEAL PLASMA

The collision integral can be expressed in terms of the spatial spectral density of fluctuations δN_a, $\delta \boldsymbol{E}$. For a spatially homogeneous plasma this expression is:

$$I_a(\boldsymbol{p},t) = -\frac{e_a}{n_a}\frac{1}{(2\pi)^3}\frac{\partial}{\partial \boldsymbol{p}}\cdot\int d\boldsymbol{k}\;\mathrm{Re}\,(\delta N_a\,\delta \boldsymbol{E})_{\boldsymbol{k},\boldsymbol{p},t}\,. \tag{53.1}$$

The equation for the fluctuations follows from (33.6) :

$$\delta N_a(\boldsymbol{k},\boldsymbol{p},t) = \delta N_a^{\text{source}}(\boldsymbol{k},\boldsymbol{p},t)$$
$$-\,e_a n_a\int_0^\infty d\tau\; e^{-\Delta\tau-i\boldsymbol{k}\cdot\boldsymbol{v}\tau}\,\delta \boldsymbol{E}(\boldsymbol{k},t-\tau)\cdot\frac{\partial f_a(\boldsymbol{p},t-\tau)}{\partial \boldsymbol{p}} \tag{53.2}$$

$$\delta \boldsymbol{E}(\boldsymbol{k},t) = -\frac{i\boldsymbol{k}}{k^2}\sum_a 4\pi e_a\int d\boldsymbol{p}\;\delta N_a(\boldsymbol{k},\boldsymbol{p},t)\,. \tag{53.3}$$

Substituting δN_a from (53.2) into (53.3), we obtain an integral equation for $\delta \boldsymbol{E}$:

$$\delta \boldsymbol{E}(\boldsymbol{k},t) - \frac{i}{k^2}\sum_a 4\pi e_a^2 n_a\int_0^\infty d\tau\int d\boldsymbol{p}\; e^{-\Delta\tau-i\boldsymbol{k}\cdot\boldsymbol{v}\tau}\,\boldsymbol{k}\cdot\frac{\partial f_a(\boldsymbol{p},t-\tau)}{\partial \boldsymbol{p}}\,\delta \boldsymbol{E}(\boldsymbol{k},t-\tau)$$
$$= -\frac{i\boldsymbol{k}}{k^2}\sum_a 4\pi e_a\int d\boldsymbol{p}\;\delta N_a^{\text{source}}(\boldsymbol{k},\boldsymbol{p},t) \equiv \delta \boldsymbol{E}^{\text{source}}(\boldsymbol{k},t) \tag{53.4}$$

We solve this equation by using perturbation theory. To zeroth order, we have

$$\delta \boldsymbol{E}^{(0)} = \delta \boldsymbol{E}^{\text{source}}\,. \tag{53.5}$$

To first order :

$$\delta \boldsymbol{E}^{(1)} = \frac{i}{k^2}\sum_a 4\pi e_a^2 n_a\int_0^\infty d\tau\int d\boldsymbol{p}\; e^{-\Delta\tau-i\boldsymbol{k}\cdot\boldsymbol{v}\tau}\,\boldsymbol{k}\cdot\frac{\partial f_a(\boldsymbol{p},t-\tau)}{\partial \boldsymbol{p}}$$
$$\cdot\,\delta \boldsymbol{E}^{\text{source}}(\boldsymbol{k},t-\tau)\,. \tag{53.6}$$

We represent the spectral density in (53.1) as a sum of two terms :

$$\left(\delta N_a\ \delta \boldsymbol{E}\right)_{\boldsymbol{k},\boldsymbol{p},t} = \left(\delta N_a\ \delta \boldsymbol{E}\right)^{\mathbf{ind}}_{\boldsymbol{k},\boldsymbol{p},t} + \left(\delta N_a^{\mathbf{source}}\ \delta \boldsymbol{E}\right)_{\boldsymbol{k},\boldsymbol{p},t} . \tag{53.7}$$

From Eq. (53.2) follows that

$$\left(\delta N_a\, \delta \boldsymbol{E}\right)^{\mathbf{ind}}_{\boldsymbol{k},\boldsymbol{p},t} = -\boldsymbol{k}\ \frac{e_a n_a}{k^2} \int_0^\infty d\tau\ e^{-\Delta\tau - i\boldsymbol{k}\cdot\boldsymbol{v}\tau} \left(\delta \boldsymbol{E}\cdot\delta \boldsymbol{E}\right)^*_{\boldsymbol{k},t,t-\tau}\ \boldsymbol{k}\cdot \frac{\partial f_a(\boldsymbol{p},t-\tau)}{\partial \boldsymbol{p}} . \tag{53.8}$$

This expression is proportional to the spectral density of the field fluctuations, hence it defines the induced contribution to (53.7).

To first order, we may write in the right-hand side of (53.8) :

$$\left(\delta \boldsymbol{E}\cdot\delta \boldsymbol{E}\right)^*_{\boldsymbol{k},t,t-\tau} = \left(\delta \boldsymbol{E}\cdot\delta \boldsymbol{E}\right)^{*\ \mathbf{source}}_{\boldsymbol{k},t,t-\tau} . \tag{53.9}$$

The spectral density of the fluctuations $\delta E^{\mathbf{source}}$ is determined from (52.9) :

$$\left(\delta \boldsymbol{E}\cdot\delta \boldsymbol{E}\right)^{*\ \mathbf{source}}_{\boldsymbol{k},t,t-\tau} = \sum_b \frac{(4\pi)^2 e_b^2 n_b}{k^2} \int d\boldsymbol{p}'\ e^{-\Delta\tau + i\boldsymbol{k}\cdot\boldsymbol{v}'\tau}\ f_b(\boldsymbol{p}',t-\tau) . \tag{53.10}$$

From (53.8) – (53.10) we obtain :

$$e_a\left(\delta N_a\ \delta E_i\right)^{\mathbf{ind}}_{\boldsymbol{k},\boldsymbol{p},t} = -\sum_b (4\pi)^2\ e_a^2 e_b^2 n_a n_b\ \frac{k_i k_j}{k^4} \cdot \int_0^\infty d\tau \int d\boldsymbol{p}'\ e^{-2\Delta\tau - i(\boldsymbol{k}\cdot\boldsymbol{v} - \boldsymbol{k}\cdot\boldsymbol{v}')\tau}\ \frac{\partial f_a(\boldsymbol{p},t-\tau)}{\partial p_j}\ f_b(\boldsymbol{p}',t-\tau). \tag{53.11}$$

We now go over to the second term of (53.7). Using (53.5) and (52.9) we find, to zeroth order :

$$\left(\delta N_a^{\mathbf{source}}\ \delta \boldsymbol{E}\right)^{(0)}_{\boldsymbol{k},\boldsymbol{p},t} = \frac{i\boldsymbol{k}}{k^2}\, 4\pi\, e_a n_a f_a(\boldsymbol{p},t) . \tag{53.12}$$

This expression does not contribute to the collision integral, as it is purely imaginary.

To first order, using (53.6), we find

$$\left(\delta N_a^{\mathbf{source}}\ \delta \boldsymbol{E}\right)^{(1)}_{\boldsymbol{k},\boldsymbol{p},t} = -\frac{i}{k^2}\sum_b 4\pi\, e_b^2 n_b \int_0^\infty d\tau \int d\boldsymbol{p}'\ e^{-\Delta\tau + i\boldsymbol{k}\cdot\boldsymbol{v}'\tau} \cdot \boldsymbol{k}\cdot\frac{\partial f_b(\boldsymbol{p}',t-\tau)}{\partial \boldsymbol{p}'} \left(\delta N_a\ \delta \boldsymbol{E}\right)^{\mathbf{source}}_{\boldsymbol{k},t,t-\tau,\boldsymbol{p}} \tag{53.13}$$

From (52.9) we obtain

$$\left(\delta N_a\ \delta \boldsymbol{E}\right)^{\mathbf{source}}_{\boldsymbol{k},\boldsymbol{p},t,t-\tau} = \frac{i\boldsymbol{k}}{k^2}\, 4\pi\, e_a n_a e^{-\Delta\tau - i\boldsymbol{k}\cdot\boldsymbol{v}\tau} f_a(\boldsymbol{p},t-\tau) . \tag{53.14}$$

Substitution into (53.13) yields :

$$e_a\left(\delta N_a^{\text{source}}\,\delta \boldsymbol{E}\right)^{(1)}_{\boldsymbol{k},\boldsymbol{p},t} = \sum_b (4\pi)^2\, e_a^2 e_b^2 n_b n_a \frac{\boldsymbol{k}}{k^2} \int_0^\infty d\tau \int d\boldsymbol{p}'\, e^{-2\Delta\tau - i(\boldsymbol{k}\cdot\boldsymbol{v} - \boldsymbol{k}\cdot\boldsymbol{v}')\tau} \cdot \boldsymbol{k}\cdot \frac{\partial f_a(\boldsymbol{p}', t-\tau)}{\partial \boldsymbol{p}'}\, f_a(\boldsymbol{p}, t-\tau). \tag{53.15}$$

Substituting (53.11) and (53.15) into (53.1) we find the collision integral: [40, 7]

$$I_a(\boldsymbol{p},t) = \sum_b \frac{2}{\pi} e_a^2 e_b^2 n_b \frac{\partial}{\partial p_i} \operatorname{Re} \int_0^\infty d\tau \int d\boldsymbol{p}'\, d\boldsymbol{k}\, \frac{k_i k_j}{k^4}\, e^{-2\Delta\tau - i(\boldsymbol{k}\cdot\boldsymbol{v} - \boldsymbol{k}\cdot\boldsymbol{v}')\tau} \left(\frac{\partial}{\partial p_j} - \frac{\partial}{\partial p'_j}\right) f_a(\boldsymbol{p}, t-\tau)\, f_b(\boldsymbol{p}', t-\tau).$$

If the retardation is neglected, and the τ-integration performed, this equation reduces to the Landau collision integral (38.3), (38.4). As in (38.4), the limits of integration on k in (63.16) are given by (38.2).

Thus, Eq. (53.16) differs from the Landau collision integral only by the inclusion of the retardation. We will see that this amounts to taking the non-ideality of the plasma into account, thus (53.16) can be called the collision integral of the nonideal plasma.

It is easily shown that the collision integral (53.16) obeys the following relations:

$$I(t) = \sum_a n_a \int d\boldsymbol{p}\; \phi_a(\boldsymbol{p})\, I_a(\boldsymbol{p},t) = 0 \quad \text{for} \quad \phi_a = 1, \boldsymbol{p} \tag{53.17}$$

which express the conservation of the number of particles and of the total momentum. Contrary to (37.10), the equality $I(t) = 0$ does not hold for $\phi_a = p^2/2m_a$:

$$\sum_a n_a \int d\boldsymbol{p}\, \frac{p^2}{2m_a} I_a = -\sum_a \sum_b \frac{e_a^2 e_b^2 n_a n_b}{\pi} \int_0^\infty d\tau \int d\boldsymbol{k}\, d\boldsymbol{p}\, d\boldsymbol{p}' \cdot e^{-2\Delta\tau} \sin(\boldsymbol{k}\cdot\boldsymbol{v}\tau - \boldsymbol{k}\cdot\boldsymbol{v}'\tau)\, k^{-4}\boldsymbol{k}\cdot\left(\frac{\partial}{\partial \boldsymbol{p}} - \frac{\partial}{\partial \boldsymbol{p}'}\right)\frac{\partial}{\partial t} f_a(\boldsymbol{p}, t-\tau)\, f_b(\boldsymbol{p}', t-\tau). \tag{53.18}$$

We now show that, to first order in the retardation, this expression can be represented as follows :

$$\sum_a n_a \int d\boldsymbol{p}\, \frac{p^2}{2m_a} I_a = -\frac{\partial}{\partial t} \frac{1}{(2\pi)^3} \int d\boldsymbol{k}\, \frac{(\delta \boldsymbol{E}\cdot\delta \boldsymbol{E})_{\boldsymbol{k},t}}{8\pi} \doteq -\frac{\partial}{\partial t} \frac{\langle \delta \boldsymbol{E}\cdot\delta \boldsymbol{E}\rangle}{8\pi} \tag{53.19}$$

From (53.5), (53.6) we obtain, to first order :

$$\left(\delta \boldsymbol{E}\cdot\delta \boldsymbol{E}\right)_{\boldsymbol{k},t} = \left(\delta \boldsymbol{E}\cdot\delta \boldsymbol{E}\right)^{\text{source}}_{\boldsymbol{k},t} + \left(\delta \boldsymbol{E}^{(1)}\cdot\delta \boldsymbol{E}^{(0)}\right)_{\boldsymbol{k},t} + \left(\delta \boldsymbol{E}^{(0)}\cdot\delta \boldsymbol{E}^{(1)}\right)_{\boldsymbol{k},t}$$

$$= \sum_a \frac{(4\pi)^2 e_a^2 n_a}{k^2} \int d\boldsymbol{p}\, f_a(\boldsymbol{p},t) + \sum_a\sum_b \frac{(4\pi)^3 e_a^2 e_b^2 n_a n_b}{k^4}$$

$$\cdot\, 2\,\mathrm{Re} \int_0^\infty d\tau \int d\boldsymbol{p}\, d\boldsymbol{p}' \sin\left[(\boldsymbol{k}\cdot\boldsymbol{v} - \boldsymbol{k}\cdot\boldsymbol{v}')\tau\right] f_a(\boldsymbol{p},t-\tau)\, \boldsymbol{k}\cdot\frac{\partial f_b(\boldsymbol{p}',t-\tau)}{\partial \boldsymbol{p}'} . \tag{53.20}$$

The first term is time-independent and describes the contribution of the self-action to the spectral density. The second term, substituted into the right-hand side of (53.19) and symmetrized, yields (53.18).

Thus, taking into account the retardation, we obtain the following energy balance for the spatially homogeneous plasma :

$$\frac{\partial}{\partial t}\left\{\sum_a n_a \int d\boldsymbol{p}\, \frac{p^2}{2m_a} f_a + \frac{1}{8\pi}\left\langle\delta \boldsymbol{E}\cdot\delta \boldsymbol{E}\right\rangle\right\} = 0 \tag{53.21}$$

which expresses the conservation of the total energy.

In equilibrium, Eq. (53.20) reduces to

$$\left(\delta \boldsymbol{E}\cdot\delta \boldsymbol{E}\right)_{\boldsymbol{k}} = \sum_a \frac{(4\pi)^2 e_a^2 n_a}{k^2} \int d\boldsymbol{p}\, f_a(\boldsymbol{p})$$

$$+ \sum_a\sum_b \frac{(4\pi)^2 e_a^2 e_b^2 n_a n_b}{k^2\, k_{\mathrm{B}} T} \int_0^\infty d\tau \int d\boldsymbol{p}\, d\boldsymbol{p}' (\boldsymbol{k}\cdot\boldsymbol{v} - \boldsymbol{k}\cdot\boldsymbol{v}') \sin\left(\boldsymbol{k}\cdot\boldsymbol{v}\tau - \boldsymbol{k}\cdot\boldsymbol{v}'\tau\right) f_a f_b \tag{53.22}$$

By integrating over τ, $\boldsymbol{p}$, $\boldsymbol{p}'$, we obtain

$$\left(\delta \boldsymbol{E}\cdot\delta \boldsymbol{E}\right)_{\boldsymbol{k}} = \sum_a \frac{(4\pi)^2 e_a^2 n_a}{k^2} - \sum_a\sum_b \frac{(4\pi)^2 e_a^2 e_b^2 n_a n_b}{k^4\, k_{\mathrm{B}} T} = \frac{(4\pi)^2 e_a^2 n_a}{k^2}\left(1 - \frac{1}{r_{\mathrm{D}}^2 k^2}\right) \tag{53.23}$$

where we use the definition of the Debye radius.

Note that this equation can also be obtained from (36.18), which can be written as :

$$\left(\delta \boldsymbol{E}\cdot\delta \boldsymbol{E}\right)_{\boldsymbol{k}} = \frac{4\pi k_{\mathrm{B}} T}{1 + r_{\mathrm{D}}^2 k^2} = \sum_a \frac{(4\pi)^2 e_a^2 n_a}{k^2} \frac{1}{1 + (r_{\mathrm{D}}^2 k^2)^{-1}} \tag{53.24}$$

and can be expanded in powers of $(r_{\mathrm{D}}^2 k^2)^{-1}$.

54. THE BALESCU-LENARD EQUATION FOR A NONIDEAL PLASMA

For the derivation of the collision integral, taking into account the polarization, it is more convenient to use, instead of (53.1), the first definition (33.5)

$$I_a(\boldsymbol{p},t) = -\frac{e_a}{n_a}\frac{1}{(2\pi)^4}\frac{\partial}{\partial \boldsymbol{p}}\cdot\int d\omega\, d\boldsymbol{k}\, \mathrm{Re}\left(\delta N_a\, \delta \boldsymbol{E}\right)_{\omega,\boldsymbol{k},\boldsymbol{p},t} . \tag{54.1}$$

In the case of nonideal plasmas we must take account of the non-stationarity of the fluctuation process.

We rewrite Eq. (35.1) as an integral equation :

$$\delta N_a(x,t) = \delta N_a^{\text{source}}(x,t) - e_a n_a \int_0^\infty d\tau\, \delta \boldsymbol{E}(\boldsymbol{r}-\boldsymbol{v}\cdot\tau, t-\tau) \cdot \frac{\partial f_a(\boldsymbol{p},t-\tau)}{\partial \boldsymbol{p}} \tag{54.2}$$

To zeroth order in the retardation this equation leads to (35.2), in which t enters as a parameter.

We note that the collision integral (54.1) is defined in terms of the spectral density of the fast fluctuations (see sect. 33). We therefore perform in (54.2) a Fourier transformation with respect to the coordinates and the fast time:

$$\delta N_a(\omega,\boldsymbol{k},\boldsymbol{p}) = \delta N_a^{\text{source}}(\omega,\boldsymbol{k},\boldsymbol{p}) - \frac{e_a n_a}{k^2} \int_0^\infty d\tau\, A(\tau)\, \boldsymbol{k}\cdot \frac{\partial f_a(\boldsymbol{p},t-\tau)}{\partial \boldsymbol{p}}\, \boldsymbol{k}\cdot\delta E(\omega,\boldsymbol{k},t-\tau). \tag{54.3}$$

For compactness, we introduced here the notation:

$$A(\tau) = \exp\left[-\Delta\tau + i(\omega - \boldsymbol{k}\cdot\boldsymbol{v})\tau\right]. \tag{54.4}$$

From the Poisson equation, we obtain a second relationship :

$$\delta \boldsymbol{E}(\omega,\boldsymbol{k},t-\tau) = -\frac{i\boldsymbol{k}}{k^2} \sum_a 4\pi e_a \int d\boldsymbol{p}\, \delta N_a(\omega,\boldsymbol{k},\boldsymbol{p},t). \tag{54.5}$$

If we used the equation for the total current we would get :

$$\left[\frac{\partial}{\partial t} - i(\omega + i\Delta)\right] \delta E(\omega,\boldsymbol{k},t) = -4\pi \sum_a e_a \int d\boldsymbol{p}\, \boldsymbol{k}\frac{\boldsymbol{k}\cdot\boldsymbol{v}}{k^2}\, \delta N_a(\omega,\boldsymbol{k},\boldsymbol{p},t) \equiv -4\pi\, \delta \boldsymbol{j}(\omega,\boldsymbol{k}). \tag{54.6}$$

Using, (54.3), we now eliminate δN_a from (54.5). As a result we obtain the following integral equation for $\delta \boldsymbol{E}(\omega,\boldsymbol{k},t)$:

$$\delta \boldsymbol{E}(\omega,\boldsymbol{k},t) - i\sum_a \frac{4\pi e_a^2 n_a}{k^2} \int_0^\infty d\tau \int d\boldsymbol{p}\, A(\tau)\, \boldsymbol{k}\cdot \frac{\partial f_a(\boldsymbol{p},t-\tau)}{\partial \boldsymbol{p}}\, \delta \boldsymbol{E}(\omega,\boldsymbol{k},t-\tau)$$
$$= -4\pi i\, \frac{\boldsymbol{k}}{k^2} \sum_a e_a \int d\boldsymbol{p}\, \delta N_a^{\text{source}}(\omega,\boldsymbol{k},\boldsymbol{p},t) \equiv \delta E^{\text{source}}(\omega,\boldsymbol{k},t). \tag{54.7}$$

For an ideal plasma, the fluctuation process defining the collision integral can be considered as homogeneous and stationary. The time dependence (through f_a) is

not essential in this approximation, and can be neglected in the evaluation of the spectral densities. For the nonideal plasma we must use Eqs (54.3) and (54.7) instead of (35.2) and (35.3). The fluctuation process is non-stationary, and this fact considerably complicates the calculations.

We have already calculated spectral densities of δN_a, δE for a non-stationary process in the derivation of the kinetic equation of a plasma in a strong high-frequency external field (sects. 41–44). In that case the problem could be reduced to a simpler one, because the non-stationary process is defined in terms of an explicit time-dependence of the external field. In the present calculation, we are bound to use a perturbation expansion in the retardation. The method of calculation developed in sections 51, 52 is inapplicable here, because the integral equation (54.7) cannot be solved explicitly.

To zeroth order in the retardation, Eq. (54.7) reduces to (35.3). We now consider the first order approximation. We perform an expansion in powers of $\tau(\partial/\partial t)$ limited to the two first terms, thus obtaining :

$$\delta \boldsymbol{E}(\omega,\boldsymbol{k},t) - i\sum_a \frac{4\pi e_a^2 n_a}{k^2}\int_0^\infty d\tau \int d\boldsymbol{p}\; A(t)\left(1-\tau\frac{\partial}{\partial t}\right)$$
$$\boldsymbol{k}\cdot\frac{\partial f_a(\boldsymbol{p},t)}{\partial \boldsymbol{p}}\;\delta E(\omega,\boldsymbol{k},t) = \delta E^{\,\text{source}}(\omega,\boldsymbol{k},t). \tag{54.8}$$

We integrate now over τ. Recalling the definition (54.4), we obtain

$$\int d\tau\; A(\tau) = \frac{i}{\omega-\boldsymbol{k}\cdot\boldsymbol{v}+i\Delta}\,,\qquad \int d\tau\;\tau A(\tau) = -i\frac{\partial}{\partial\omega}\,\frac{1}{\omega-\boldsymbol{k}\cdot\boldsymbol{v}+i\Delta}\,. \tag{54.9}$$

Using the expression (35.5) for the dielectric constant, we obtain, to first order:

$$\varepsilon(\omega,\boldsymbol{k})\;\;\delta E(\omega,\boldsymbol{k},t) = \delta E^{\,\text{source}} - i\,\frac{\partial}{\partial t}\left[\frac{\partial\varepsilon(\omega,\boldsymbol{k})}{\partial\omega}\;\frac{\delta E^{\,\text{source}}}{\varepsilon}\right]. \tag{54.10}$$

Let us consider the energy balance equation, taking into account the retardation. Using Eq. (54.6) we obtain

$$\frac{\partial}{\partial t}\,\frac{(\delta\boldsymbol{E}\cdot\delta\boldsymbol{E})_{\omega,\boldsymbol{k},t}}{8\pi} = -\,\mathrm{Re}\,(\delta\boldsymbol{j}\cdot\delta\boldsymbol{E})_{\omega,\boldsymbol{k},t}\,. \tag{54.11}$$

We recall that the kinetic equation (26.3) leads to the balance of kinetic energy which, in the homogeneous case, reduces to

$$\frac{\partial}{\partial t}\sum_a n_a\int d\boldsymbol{p}\,\frac{p^2}{2m_a}\,f_a = \langle\delta\boldsymbol{j}\cdot\delta\boldsymbol{E}\rangle \equiv \sum_a n_a\int d\boldsymbol{p}\,\frac{p^2}{2m_a}\,I_a\,. \tag{54.12}$$

For an ideal plasma, the collision integral possesses the property (37.10), hence

$$\langle\delta\boldsymbol{j}\cdot\delta\boldsymbol{E}\rangle = (2\pi)^{-4}\int d\omega\;d\boldsymbol{k}\;\mathrm{Re}\,(\delta\boldsymbol{j}\cdot\delta\boldsymbol{E})_{\omega\boldsymbol{k}} = 0 \tag{54.13}$$

Thus, for an ideal plasma,

$$\frac{\partial}{\partial t}\langle \delta E \cdot \delta E \rangle = 0 \,. \tag{54.14}$$

As a result, the contribution of the electric field to the balance of energy vanishes. This agrees with the results of the previous section.

We now consider the right-hand side of (54.11) in the case in which the retardation is taken into account. For convenience, we add to the right-hand side the term $\operatorname{Re} i\omega(\delta E \cdot \delta E)_{\omega,\boldsymbol{k},t}$ and consider the expression

$$-\operatorname{Re}(\delta \boldsymbol{j}\cdot\delta \boldsymbol{E})_{\omega,\boldsymbol{k},t} + \operatorname{Re} i\omega \frac{1}{4\pi}(\delta \boldsymbol{E}\cdot\delta \boldsymbol{E})_{\omega,\boldsymbol{k},t}$$
$$\equiv \operatorname{Re}\left[\left(\frac{i\omega}{4\pi}\delta \boldsymbol{E} - \delta \boldsymbol{j}\right)\cdot\delta \boldsymbol{E}\right]_{\omega,\boldsymbol{k},t} . \tag{54.15}$$

From (54.3) we obtain

$$\delta \boldsymbol{j}(\omega,\boldsymbol{k},t) = \delta \boldsymbol{j}^{\,\text{source}} - \sum_a \frac{e_a^2 n_a}{k^2}\int_0^\infty d\tau \int d\boldsymbol{p}\; \boldsymbol{k}\cdot\boldsymbol{v}\, A(\tau)\, \boldsymbol{k}\cdot\frac{\partial f_a(\boldsymbol{p},t-\tau)}{\partial \boldsymbol{p}}\,\delta \boldsymbol{E}(\omega,\boldsymbol{k},t-\tau) \,. \tag{54.16}$$

From this equation and from (54.7) we obtain the combination of functions entering (54.15):

$$\frac{1}{4\pi} i(\omega+i\Delta)\,\delta \boldsymbol{E} - \delta \boldsymbol{j} = \frac{1}{4\pi} i(\omega+i\Delta)\,\delta \boldsymbol{E}^{\text{source}} - \delta \boldsymbol{j}^{\,\text{source}}$$
$$-\sum_a \frac{e_a^2 n_a}{k^2}\int_0^\infty d\tau \int d\boldsymbol{p}\left[(\omega+i\Delta) - \boldsymbol{k}\cdot\boldsymbol{v}\right] A(\tau)\, \boldsymbol{k}\cdot\frac{\partial f_a(\boldsymbol{p},t-\tau)}{\partial \boldsymbol{p}}\,\delta \boldsymbol{E}(\omega,\boldsymbol{k},t-\tau) \,. \tag{54.17}$$

We integrate the right-hand side by parts (recalling Eq. (54.4)):

$$\frac{1}{4\pi} i(\omega+i\Delta)\,\delta \boldsymbol{E} - \delta \boldsymbol{j} = \frac{1}{4\pi} i(\omega+i\Delta)\,\delta \boldsymbol{E}^{\text{source}} - \delta \boldsymbol{j}^{\,\text{source}}$$
$$-i\sum_a \frac{e_a^2 n_a}{k^2}\int_0^\infty d\tau \int d\boldsymbol{p}\; A(\tau)\frac{\partial}{\partial \tau}\, \boldsymbol{k}\cdot\frac{\partial f_a(\boldsymbol{p},t-\tau)}{\partial \boldsymbol{p}}\,\delta \boldsymbol{E}(\omega,\boldsymbol{k},t-\tau) \,. \tag{54.18}$$

In this expression we change the τ-derivative into a t-derivative (as $(\partial/\partial\tau) = -(\partial/\partial t)$). As the integral term is proportional to $\partial/\partial t$ the retardation can be neglected. Using Eqs (54.9) and the expression (35.5) for the dielectric constant, we find

$$\frac{1}{4\pi} i(\omega+i\Delta)\,\delta \boldsymbol{E} - \delta \boldsymbol{j} = -\left[\left(\frac{\partial}{\partial t} - i(\omega+i\Delta)\right)\frac{\delta \boldsymbol{E}^{\text{source}}}{4\pi} + \delta \boldsymbol{j}^{\text{source}}\right]$$
$$+\frac{1}{4\pi}\frac{\partial}{\partial t}\left(\frac{\delta \boldsymbol{E}^{\text{source}}}{\varepsilon}\right) .$$

As a result of the equation for the total current, the bracketed term in this

expression vanishes, and we are left with

$$\frac{i(\omega+i\Delta)}{4\pi}\,\delta E - \delta j = \frac{1}{4\pi}\,\frac{\partial}{\partial t}\left(\frac{\delta E^{\text{source}}}{\varepsilon}\right) . \qquad (54.19)$$

Substituting this expression into (54.15) we find

$$-\ \mathrm{Re}\left(\delta \boldsymbol{j}\cdot\delta \boldsymbol{E}\right)_{\omega,\boldsymbol{k},t} = \frac{1}{8\pi}\,\frac{\partial}{\partial t}\left[\frac{(\delta\boldsymbol{E}\cdot\delta\boldsymbol{E})^{\text{source}}_{\omega,\boldsymbol{k},t}}{|\varepsilon(\omega,\boldsymbol{k},t)|^2}\right] \qquad (54.20)$$

Here $(\delta\boldsymbol{E}\cdot\delta\boldsymbol{E})^{\text{source}}_{\omega,\boldsymbol{k},t}$ is the spectral density of the fluctuations δE^{source} to zeroth order in the retardation. It is defined by (34.14), i.e.

$$\left(\delta\boldsymbol{E}\cdot\delta\boldsymbol{E}\right)^{\text{source}}_{\omega,\boldsymbol{k},t} = \sum_a \frac{(4\pi)^2\,e_a^2 n_a}{k^2}\int d\boldsymbol{p}\;2\pi\,\delta(\omega-\boldsymbol{k}\cdot\boldsymbol{v})\,f_a \qquad (54.21)$$

Thus, from the balance of kinetic energy (54.12) and the balance of the field energy (54.11), (54.20) follows again the conservation of the total energy (53.21), but now we must use instead of (53.22), the more general equation

$$\left(\delta\boldsymbol{E}\cdot\delta\boldsymbol{E}\right)_{\boldsymbol{k},t} = \sum_a \frac{(4\pi)^2\,e_a^2 n_a}{k^2}\int d\boldsymbol{p}\;\frac{f_a(\boldsymbol{p},t)}{|\varepsilon(\boldsymbol{k}\cdot\boldsymbol{v},\boldsymbol{k})|^2} \qquad (54.22)$$

which follows from (54.21). Equation (53.23) is a particular case of (54.22). To see this, we must expand the polarization contribution, keep terms up to the first order and introduce the local equilibrium approximation.

We now consider the collision integral taking into account the polarization and the retardation.

As in sect. 53, Eq. (53.7), the spectral density of δN_a, δE can be represented as a sum of two terms; hence

$$\begin{aligned} I_a &= I_a^{\text{ind}} - \frac{e_a}{n_a}(2\pi)^{-4}\,\frac{\partial}{\partial\boldsymbol{p}}\cdot\int d\omega\,d\boldsymbol{k}\;\mathrm{Re}\left(\delta N_a^{\text{source}}\,\delta\boldsymbol{E}\right)_{\omega,\boldsymbol{k},\boldsymbol{p},t} \\ &\equiv I_a^{\text{ind}} + I_a^{\text{source}} . \end{aligned} \qquad (54.23)$$

From (54.3) follows the expression of the induced part of the collision integral [7, 40]

$$\begin{aligned} I_a^{\text{ind}} = \frac{e_a^2}{(2\pi)^4}\,\frac{\partial}{\partial\boldsymbol{p}}\cdot\mathrm{Re}\int_0^\infty d\tau\int d\omega\,d\boldsymbol{k}\,\frac{\boldsymbol{k}}{k^2}\,\exp\left[-\Delta\tau + i(\omega-\boldsymbol{k}\cdot\boldsymbol{v})\tau\right] \\ \cdot\left(\delta\boldsymbol{E}\cdot\delta\boldsymbol{E}\right)^{*}_{\omega,\boldsymbol{k},t,t-\tau}\;\boldsymbol{k}\cdot\frac{\partial f_a(\boldsymbol{p},t-\tau)}{\partial\boldsymbol{p}} . \end{aligned} \qquad (54.24)$$

The spectral density of the field fluctuations appearing here can be expressed as follows, to first order in the retardation :

$$\left(\delta\boldsymbol{E}\cdot\delta\boldsymbol{E}\right)^{*}_{\omega,\boldsymbol{k},t-\tau} = \left(\delta\boldsymbol{E}\cdot\delta\boldsymbol{E}\right)^{*}_{\omega,\boldsymbol{k},t} - \tau\left(\delta\boldsymbol{E}\cdot\frac{\partial}{\partial t}\delta\boldsymbol{E}\right)^{*}_{\omega,\boldsymbol{k},t}$$

$$= (1+i\omega\tau)\left(\delta\boldsymbol{E}\cdot\delta\boldsymbol{E}\right)_{\omega,\boldsymbol{k},t} + \tau 4\pi\left(\delta\boldsymbol{j}\cdot\delta\boldsymbol{E}\right)_{\omega,\boldsymbol{k},t} \tag{54.25}$$

We used here the total current equation for $\delta\boldsymbol{E}(\omega,\boldsymbol{k},t)$. By means of Eqs (54.10), (54.16) the spectral densities appearing here can be expressed in terms of the spectral densities of the source fluctuations (see (52.12)). We may thus obtain an explicit expression for I_a^{ind}. We shall not develop these calculations here, as we shall only need Eq. (54.24) in our further discussions (sect. 57).

Consider now the second term on the right-hand side of (54.23). Using (54.10), we find

$$I_a^{\mathrm{source}} = -\frac{e_a}{n_a}(2\pi)^{-4}\frac{\partial}{\partial\boldsymbol{p}}$$

$$\cdot\int d\omega\, d\boldsymbol{k}\ \mathrm{Re}\left\{\frac{1}{\varepsilon^*}\left[1+i\frac{\partial\varepsilon^*}{\partial\omega}\frac{\partial}{\partial t}\frac{1}{\varepsilon^*}\right]\left(\delta N_a\ \delta\boldsymbol{E}\right)^{\mathrm{source}}_{\omega,\boldsymbol{k},\boldsymbol{p}}\right.$$

$$\left. + i\frac{\partial\varepsilon^*}{\partial\omega}\frac{1}{\varepsilon^{*2}}\left(\delta N_a\ \frac{\partial\delta\boldsymbol{E}}{\partial t}\right)^{\mathrm{source}}_{\omega,\boldsymbol{k},\boldsymbol{p}}\right\} \tag{54.26}$$

The spectral density $(\delta N_a\ \delta\boldsymbol{E})^{\mathrm{source}}_{\omega,\boldsymbol{k},\boldsymbol{p}}$ is defined by (52.12). The second spectral density can be obtained from Eqs (52.11) and $(52.12)_1$:

$$\left(\delta N_a\ \frac{\partial\delta\boldsymbol{E}}{\partial t}\right)^{\mathrm{source}}_{\omega,\boldsymbol{k},\boldsymbol{p}} = \frac{\boldsymbol{k}}{k^2}4\pi e_a n_a\cdot 2\,\mathrm{Re}\int_0^\infty d\tau\,(\omega-\boldsymbol{k}\cdot\boldsymbol{v})$$

$$\cdot\exp\left[-\Delta\tau + i(\omega-\boldsymbol{k}\cdot\boldsymbol{v})\tau\right] f_a(\boldsymbol{p},t-\tau)\ . \tag{54.27}$$

Combining all these results, we obtain an explicit expression for I_a^{source}.

The results (54.24) and (54.26) define the collision integral for the nonideal plasma taking account of the polarization. To zeroth order in the retardation it reduces to the Balescu-Lenard collision integral (in the form (37.2)).

This collision integral for nonideal plasmas possesses the same properties (53.17), (53.19) as the Landau collision integral for nonideal plasmas. The only difference is that the field energy is now expressed by (54.22) in which the polarization of the plasma is fully taken into account.

55. THE COLLISION INTEGRAL OF THE NONIDEAL PLASMA, TAKING INTO ACCOUNT THE AVERAGED DYNAMICAL POLARIZATION

We consider the collision integral for a nonideal plasma in which the dynamical polarization is treated approximately by means of an effective potential. The latter is defined by Eqs (47.2), (47.6). If these formulae are evaluated in local equilibrium, we obtain Eqs (47.3), (47.4) :

$$\tilde{\nu}_{ab}(\boldsymbol{k}) = \nu_{ab}(\boldsymbol{k}) \frac{r_{\mathbf{D}}^2 k^2}{1 + r_{\mathbf{D}}^2 k^2} , \quad \tilde{\Phi}_{ab}(\boldsymbol{r}) = \frac{e_a e_b}{r} e^{-r/r_{\mathbf{D}}} , \tag{55.1}$$

where

$$\nu_{ab}(\boldsymbol{k}) = 4\pi e_a e_b / k^2 .$$

In this approximation, we may use the results of section 53 for the collision integral. We simply replace, in the evaluation of the spectral densities, the Fourier transform of the Coulomb potential $\nu_{ab}(\boldsymbol{k})$ by the expression (55.1) for $\tilde{\nu}_{ab}(\boldsymbol{k})$. Thus, for the collision integral we obtain the result (53.16) with the change

$$\frac{k_i k_j}{k^4} \longrightarrow \frac{k_i k_j}{k^4} \frac{r_{\mathbf{D}}^2 k^2}{1 + r_{\mathbf{D}}^2 k^2} . \tag{55.2}$$

We now study the properties of the new collision integral. The properties (53.17) are obviously unchanged, but (53.16) is modified to

$$\sum_a n_a \int d\boldsymbol{p} \frac{p^2}{2m_a} I_a = - \sum_a \sum_b \frac{e_a^2 e_b^2 n_a n_b}{\pi} \int_0^\infty d\tau \int d\boldsymbol{k}\, d\boldsymbol{p}\, d\boldsymbol{p}'\, e^{-2\Delta\tau} \sin(\boldsymbol{k}\cdot\boldsymbol{v}\tau - \boldsymbol{k}\cdot\boldsymbol{v}'\tau)$$
$$\cdot \frac{r_{\mathbf{D}}^2 k^2}{k^4 (1 + r_{\mathbf{D}}^2 k^2)} \boldsymbol{k}\cdot\left(\frac{\partial}{\partial \boldsymbol{p}} - \frac{\partial}{\partial \boldsymbol{p}'}\right) \frac{\partial}{\partial t} f_a(\boldsymbol{p}, t-\tau) f_b(\boldsymbol{p}', t-\tau) . \tag{55.3}$$

In the case of local equilibrium, and to first order in the retardation :

$$\sum_a n_a \int d\boldsymbol{p} \frac{p^2}{2m_a} I_a = - \frac{\partial}{\partial t} \frac{1}{(2\pi)^3} \sum_a \sum_b \frac{n_a n_b}{2} \int d\boldsymbol{k} \frac{4\pi e_a e_b}{k^2} g_{ab}(\boldsymbol{k}) , \tag{55.4}$$

where

$$g_{ab}(\boldsymbol{k}) = - \frac{e_a e_b}{\sum_c e_c^2 n_c} \frac{1}{1 + r_{\mathbf{D}}^2 k^2} \tag{55.5}$$

is the Fourier transform of the spatial correlation function of the charged particles (see (36.22)).

Let us transform (55.4) to the coordinate representation. Using the integral (36.23) we obtain

$$\sum_a n_a \int d\boldsymbol{p} \frac{p^2}{2m_a} I_a = - \frac{\partial}{\partial t} \sum_a \sum_b \frac{n_a n_b}{2} \int d\boldsymbol{r} \frac{e_a e_b}{r} g_{ab}(\boldsymbol{r}) \tag{55.6}$$

where

$$g_{ab}(\boldsymbol{r}) = - \frac{e_a e_b}{k_{\mathbf{B}} T} \frac{e^{-r/r_{\mathbf{D}}}}{r} \tag{55.7}$$

is the spatial correlation function for a Coulomb plasma in the Debye approximation. We recall that we are considering here local equilibrium; hence the temperature T in (55.6) depends on the position and on time.

From (54.12) and (55.6) follows the conservation of energy :

$$\partial U / \partial t = 0$$

$$U = \sum_a n_a \int d\boldsymbol{p} \frac{p^2}{2m_a} f_a + \tfrac{1}{2} \sum_a \sum_b \int d\boldsymbol{r} \frac{e_a e_b}{r} g_{ab}(\boldsymbol{r}) . \tag{55.8}$$

If we include in (55.8) the self-energy, defined according to (53.23) by

$$\frac{1}{(2\pi)^3}\cdot\frac{1}{8\pi}\int d\boldsymbol{k}\sum_a \frac{(4\pi)^2 e_a^2 n_a}{k^2} \tag{55.9}$$

then Eq. (55.4) becomes

$$\sum_a n_a \int d\boldsymbol{p}\, \frac{p^2}{2m_a}\, I_a$$

$$= -\frac{\partial}{\partial t}\frac{1}{(2\pi)^3}\int d\boldsymbol{k}\, \frac{1}{8\pi}\frac{4\pi\, k_B T}{1 + r_D^2 k^2} = -\frac{\partial}{\partial t}\frac{\langle \delta \boldsymbol{E}\cdot \delta \boldsymbol{E}\rangle}{8\pi}\ . \tag{55.10}$$

We used here Eq. (53.24) for the spectral function of the field fluctuations in the local equilibrium. From (54.12) and (55.10) follows the conservation of the energy in the form (53.21).

We note that for a spatially homogeneous plasma, the two-particle distribution function (integrated over the momenta) is given by

$$f_{ab}(\boldsymbol{r}) = 1 + g_{ab}(\boldsymbol{r}) \tag{55.11}$$

hence the expression of the internal energy can be written in the form :

$$U = \sum_a n_a \int d\boldsymbol{p}\, \frac{p^2}{2m_a} f_a + \tfrac{1}{2}\sum_a\sum_b n_a n_b \int d\boldsymbol{r}\, \frac{e_a e_b}{r} f_{ab}(\boldsymbol{r})\ . \tag{55.12}$$

In the general case, away from equilibrium, the Fourier transform of the correlation function is given by

$$g_{ab}(\boldsymbol{k},\boldsymbol{p},\boldsymbol{p}',t) = i\tilde{v}_{ab}(\boldsymbol{k})\int_0^\infty d\tau\, \exp\left[-2\Delta\tau - i\boldsymbol{k}(\boldsymbol{v}-\boldsymbol{v}')\tau\right]$$

$$\boldsymbol{k}\cdot\left(\frac{\partial}{\partial \boldsymbol{p}} - \frac{\partial}{\partial \boldsymbol{p}'}\right) f_a(\boldsymbol{p},t-\tau)\, f_b(\boldsymbol{p}',t-\tau)\ . \tag{55.13}$$

This equation follows from (53.2) , (53.5) and (5-.6), if we also use

$$\left(\delta N_a\, \delta N_b\right)_{\boldsymbol{k},\boldsymbol{p},\boldsymbol{p}',t} = \delta_{ab}\,\delta(\boldsymbol{p}-\boldsymbol{p}')\, n_a f_a + n_a n_b\, g_{ab}(\boldsymbol{k},\boldsymbol{p},\boldsymbol{p}',t) \tag{55.14}$$

Using this expression, the collision integral (53.1) can be transformed to

$$I_a(\boldsymbol{p},t) = \sum_b n_b \int d\boldsymbol{r}\, d\boldsymbol{p}\, d\boldsymbol{p}'\left(\frac{\partial}{\partial \boldsymbol{r}}\frac{e_a e_b}{r}\right)\cdot\frac{\partial}{\partial \boldsymbol{p}}\, g_{ab}(\boldsymbol{r},\boldsymbol{p},\boldsymbol{p}',t)$$

$$= \frac{1}{(2\pi)^3}\sum_b n_b \frac{\partial}{\partial \boldsymbol{p}}\cdot\int d\boldsymbol{k}\, d\boldsymbol{p}'\, \boldsymbol{k}\, \frac{4\pi e_a e_b}{k^2}\, \mathrm{Im}\, g_{ab}(\boldsymbol{k},\boldsymbol{p},\boldsymbol{p}',t)\ . \tag{55.15}$$

From Eqs (55.13) , (55.15) follows that, to first order in the retardation, the collision integral can be split as follows :

$$I_a(p,t) = I_{a,1}(p,t) + I_{a,2}(p,t) \tag{55.16}$$

Here $I_{a,1}$ is the collision integral for the ideal plasma (zeroth order in the retardation). The second term determines the contribution of the correlations to the non-dissipative characteristics : [7 , 40]

$$I_{a_1} = \frac{1}{8\pi^2} \sum_b n_b \frac{\partial}{\partial p_i} \int d\boldsymbol{k} \; d\boldsymbol{p}' \; k_i k_j \, \nu_{ab}(\boldsymbol{k}) \, \tilde{\nu}_{ab}(\boldsymbol{k}) \; \delta(\boldsymbol{k}\cdot\boldsymbol{v} - \boldsymbol{k}\cdot\boldsymbol{v}') \left(\frac{\partial}{\partial p_j} - \frac{\partial}{\partial p'_j}\right) f_a(\boldsymbol{p}, t) \; f_b(\boldsymbol{p}', t) \tag{55.17}$$

$$I_{a_2} = -\frac{1}{(2\pi)^3} \sum_b n_b \frac{\partial}{\partial p_i} \int_0^\infty d\tau \int d\boldsymbol{k} \; d\boldsymbol{p}' \; k_i k_j \, \nu_{ab}(\boldsymbol{k}) \; \tilde{\nu}_{ab}(\boldsymbol{k}) \; \tau \cos(\boldsymbol{k}\cdot\boldsymbol{v}\tau - \boldsymbol{k}\cdot\boldsymbol{v}'\tau)\left(\frac{\partial}{\partial p_j} - \frac{\partial}{\partial p'_j}\right)\frac{\partial}{\partial t} f_a(\boldsymbol{p}, t) \; f_b(\boldsymbol{p}', t). \tag{55.18}$$

The functions $\nu_{ab}(\boldsymbol{k})$, $\tilde{\nu}_{ab}(\boldsymbol{k})$ are defined by (55.1).

Equations (55.16) – (55.18) correspond to Eqs (13.11) – (13.13) for gases. The latter equations were obtained in the weak coupling approximation, whereas the present formulae take account of the average polarization of the plasma, through the effective potential $\tilde{\Phi}_{ab}(\boldsymbol{r})$.

56. BOLTZMANN KINETIC EQUATION FOR A NONIDEAL PLASMA

In sections 37 and 54 we studied the kinetic equations for ideal and nonideal plasmas in the polarization approximation, i.e. to first order in the plasma parameter. In this approximation, the contribution of the interactions for short distances is incorrect. This causes, in particular, the divergence of the Balescu-Lenard equation for large wavevectors k (i.e. for short distances). It is therefore necessary to cut off the integration range at some value $k_{\mathbf{max}}$ (see sect. 37). As the quantity $k_{\mathbf{max}}$ enters the collision integral only through its logarithm, the kinetic coefficients are not very sensitive to the exact choice of the cut-off.

For the nonideal plasmas the situation is different. The short-range contribution of the interactions to the non-dissipative quantities is important; hence the polarization approximation is insufficient. We need a kinetic equation which accounts correctly for both long- and short-range contributions of the interactions.

Note that the binary collision approximation leading to the Boltzmann equation (sects 10, 12) can be used for plasmas as well for the short distance contributions ($r \ll r_{\mathrm{D}}$), when the collective effects are unimportant. However, because of the slow decrease of the Coulomb interactions with distance the Boltzmann collision integral diverges for small k (i.e. for large distances).

We shall attempt to combine these two conflicting approximations.

The simplest solution of this problem consists of combining the following three collision integrals: Boltzmann: I_a^{B} ; Landau: I_a^{L}, and Balescu-Lenard: I_a^{BL}:

$$I_a = I_a^{\mathbf{B}} - I_a^{\mathbf{L}} + I_a^{\mathbf{BL}} \ . \tag{56.1}$$

When this expression is applied to the ideal plasma, the Landau integral compensates the divergence of the Boltzmann integral at small k, and the divergence of the Balescu-Lenard integral at large k. Such a generalization of the collision integral is appealing by its simplicity; for nonideal plasmas it is however incorrect, as we shall see.

To first order in the retardation, the Boltzmann collision integral for a nonideal plasma can be found directly from Eq. (12.5) :

$$I_a^{\mathbf{B}} = \sum_b n_b \int d\boldsymbol{r}' \ d\boldsymbol{p}' \ \frac{\partial \Phi_{ab}}{\partial \boldsymbol{r}} \cdot \frac{\partial}{\partial \boldsymbol{p}} f_a(\boldsymbol{P}(-\infty), t) \ f_b(\boldsymbol{P}(-\infty), t)$$

$$- \sum_b n_b \int_0^{\infty} d\tau \int d\boldsymbol{r}' \, d\boldsymbol{p}' \ \frac{\partial \phi_{ab}}{\partial \boldsymbol{r}} \cdot \frac{\partial}{\partial \boldsymbol{p}} \tau \frac{\partial^2}{\partial t \ \partial \tau} f_a(\boldsymbol{P}(-\tau), t) \ f_b(\boldsymbol{P}(-\tau), t) \tag{56.2}$$

where

$$\Phi_{ab}(\boldsymbol{r}) = \frac{e_a e_b}{r} \ . \tag{56.3}$$

As in the case of a gas, the second term in (56.2) provides the contribution of the interactions to the non-dissipative properties of the plasma. The Landau collision integral $I_a^{\mathbf{L}}$ is given by (53.16) and the Balescu-Lenard integral by Eqs (54.24), (54.26) [or (55.17) , (55.18) in the averaged dynamical polarization approximation].

We consider the correction to the plasma energy, due to the nonideality: it is determined by the correlation function $g_{ab}(\boldsymbol{r})$. In the Boltzmann approximation:

$$g_{ab}^{\mathbf{B}}(\boldsymbol{r}) = \exp\left[-\frac{e_a e_b}{r \, k_{\mathbf{B}} T}\right] - 1 \ . \tag{56.4}$$

In the Landau approximation, which amounts to a perturbation expansion, we get from (56.4):

$$g_{ab}^{\mathbf{L}}(\boldsymbol{r}) = -\frac{e_a e_b}{r \, k_{\mathbf{B}} T} \ . \tag{56.5}$$

Finally, in the Balescu-Lenard approximation we obtain the Debye correlation function:

$$g_{ab}^{\mathbf{BL}}(\boldsymbol{r}) = -\frac{e_a e_b}{k_{\mathbf{B}} T r} e^{-r/r_{\mathbf{D}}} \ . \tag{56.6}$$

Because of the structure of the collision integral (56.1), the contribution of the interactions to the plasma energy will be defined by the correlation function:

$$g_{ab}(\boldsymbol{r}) = \exp\left[-\frac{e_a e_b}{k_{\mathbf{B}} T \, r}\right] - 1 + \frac{e_a e_b}{k_{\mathbf{B}} T r} - \frac{e_a e_b}{k_{\mathbf{B}} T r} e^{-r/r_{\mathbf{D}}} \ . \tag{56.7}$$

It is clear that at large distances the Landau term compensates the Boltzmann term only to first order in the interactions. As a result, the correlation function

decreases at large distances as r^{-2}, and not exponentially, as would be expected.

There exist other methods for obtaining a collision integral free from divergences. One of them consists of replacing the Coulomb potential by the Debye potential:

$$\Phi_{ab} = \frac{e_a e_b}{r} \longrightarrow \frac{e_a e_b}{r} e^{-r/r_\mathrm{D}} \equiv \Phi^\mathrm{D}_{ab} . \tag{56.8}$$

This approximation also yields incorrect values for the thermodynamic quantities. For instance, one obtains for the internal energy the expression :

$$\frac{1}{2} \sum_a \sum_b n_a n_b \int d\boldsymbol{r}\ \Phi^\mathrm{D}_{ab}(\boldsymbol{r})\ \Phi^\mathrm{D}_{ab}(\boldsymbol{r}) \tag{56.9}$$

which does not coincide with the equilibrium value (60.11).

Such difficulties are due to the fact that the assumption (56.8) implies that the equilibrium correlations are reached prior to the equilibrium momentum distribution. There is, however, no justification in assuming that the polarization is static.

The collision integral of a nonideal plasma, in which the interactions are correctly treated at both short and long distances, is exceedingly complicated. We shall, therefore, consider here a simplified description by means of an effective potential. We recall that the latter is defined by Eqs (47.2) , (47.6). For simplicity, we assume that the function f_a entering the definition of $\tilde{\Phi}_{ab}$ is the local Maxwell distribution. The functions $\tilde{\nu}_{ab}(\boldsymbol{k})$ and $\tilde{\Phi}_{ab}(\boldsymbol{r})$ are then given by (55.1).

As we know, in this approximation the contribution of the interactions at large distances is given by (53.16), combined with (55.2), i.e. :

$$I_a = \sum_b \frac{2}{\pi} e_a^2 e_b^2 n_b \frac{\partial}{\partial p_i} \int_0^\infty d\tau \int d\boldsymbol{k}\ d\boldsymbol{p}'\ \frac{k_i k_j}{k^2} \frac{r_\mathrm{D}^2}{1 + r_\mathrm{D}^2 k^2}$$

$$\exp\left[-2\Delta\tau - i(\boldsymbol{k}\cdot\boldsymbol{v} - \boldsymbol{k}\cdot\boldsymbol{v}')\tau\right]\left(\frac{\partial}{\partial p_j} - \frac{\partial}{\partial p_j'}\right) f_a(\boldsymbol{p}, t-\tau)\, f_b(\boldsymbol{p}', t-\tau) . \tag{56.10}$$

To first order in the retardation, this expression can be represented in the form (55.16) – (55.18).

In order to obtain the contribution of the interactions at short distances, we proceed as follows. For the function $f_{ab}(x, x', t)$ we use an equation analogous to Eq. (10.2) for the gas. In this equation, we use the effective potential $\tilde{\Phi}_{ab}(r)$ instead of $\Phi_{ab}(r)$. This implies that the function f_{ab} is determined by the averaged dynamical polarization. Noting that for spatially homogeneous systems we may use f_{ab} instead of g_{ab} in the collision integral we obtain the following closed set of equations for the functions f_a, f_b :

$$\frac{\partial f_a}{\partial t} = \sum_b n_b \int dx'\ \frac{\partial \Phi_{ab}}{\partial \boldsymbol{r}} \cdot \frac{\partial}{\partial \boldsymbol{p}} f_{ab} \equiv I_a \tag{56.11}$$

$$\left(\frac{\partial}{\partial t} + \boldsymbol{v} \cdot \frac{\partial}{\partial \boldsymbol{r}} + \boldsymbol{v}' \cdot \frac{\partial}{\partial \boldsymbol{r}'} - \frac{\partial \tilde{\Phi}_{ab}}{\partial \boldsymbol{r}} \cdot \frac{\partial}{\partial \boldsymbol{p}} - \frac{\partial \tilde{\Phi}_{ab}}{\partial \boldsymbol{r}'} \cdot \frac{\partial}{\partial \boldsymbol{p}'}\right) f_{ab}(x, x', t)$$

$$= \frac{\partial}{\partial t} f_a(\boldsymbol{p}, t)\, f_b(\boldsymbol{p}', t) \,. \tag{56.12}$$

It is important to note that the effective potential only enters the equation for f_{ab}.

Because of the analogy with (10.2), we can immediately write down the solution of (56.12), using Eq. (12.4). To first order in the retardation, the solution of (56.12) is thus : [7]

$$f_{ab} = f_a(\tilde{\boldsymbol{P}}(-\infty), t)\, f_b(\tilde{\boldsymbol{P}}(-\infty), t)$$

$$- \frac{\partial}{\partial t} \int_0^\infty d\tau\, \tau\, \frac{d}{d\tau} f_a(\tilde{\boldsymbol{P}}(-\tau), t)\, f_b(\tilde{\boldsymbol{P}}'(-\tau), t) \,. \tag{56.13}$$

Here $\tilde{\boldsymbol{P}}(-\tau)$, $\tilde{\boldsymbol{P}}'(-\tau)$ are the 'initial' (at time $t-\tau$) values of the momenta of the two particles, interacting through the potential $\tilde{\Phi}_{ab}(r)$.

From (56.11) , (56.13) follows that the collision integral can again be represented in the form [7]

$$I_a(\boldsymbol{p}, t) = I_{a_1}(\boldsymbol{p}, t) + I_{a_2}(\boldsymbol{p}, t) \tag{56.14}$$

where

$$I_{a_1}(\boldsymbol{p}, t) = \sum_b n_b \int d\boldsymbol{r}'\, d\boldsymbol{p}'\, \frac{\partial \Phi_{ab}}{\partial \boldsymbol{r}} \cdot \frac{\partial}{\partial \boldsymbol{p}} f_a(\tilde{\boldsymbol{P}}(-\infty), t)\, f_a(\tilde{\boldsymbol{P}}'(-\infty), t) \tag{56.15}$$

$$I_{a_2}(\boldsymbol{p}, t) = - \frac{\partial}{\partial t} \sum_b n_b \int_0^\infty d\tau \int d\boldsymbol{r}'\, d\boldsymbol{p}'\, \frac{\partial \Phi_{ab}}{\partial \boldsymbol{r}}$$

$$\cdot \frac{\partial}{\partial \boldsymbol{p}}\, \tau\, \frac{\partial}{\partial \tau} f_a(\tilde{\boldsymbol{P}}(-\tau), t)\, f_b(\tilde{\boldsymbol{P}}'(-\tau), t) \,. \tag{56.16}$$

These expressions are analogous to Eqs (10.5) and (12.6) for gases, but Eq. (56.15) differs significantly from the Boltzmann collision integral. Indeed, through the effective potential this term accounts as well for the pair collisions (at short distances) as for collective interactions (at large distances).

The collision integral (56.14) for nonideal plasmas has the properties

$$\sum_a n_a \int d\boldsymbol{p}\, \phi_a I_a = 0 \quad \text{for} \quad \phi_a = 1, \boldsymbol{p} \tag{56.17}$$

which correspond to the conservation of the number of particles and of the momentum. For $\phi_a = p^2/2m_a$ we obtain the following result, analogous to (12.15):

$$\sum_a n_a \int d\boldsymbol{p}\, \frac{p^2}{2m_a} I_a = - \frac{\partial}{\partial t} \frac{1}{2} \sum_a \sum_b n_a n_b \int d\boldsymbol{r}\, d\boldsymbol{p}\, d\boldsymbol{p}'\, \Phi_{ab}(\boldsymbol{r})$$

$$\cdot f_a(\tilde{\boldsymbol{P}}(-\infty), t)\, f_b(\tilde{\boldsymbol{P}}'(-\infty), t) \,. \tag{56.18}$$

This result ensures the conservation of the internal energy, U, including the interaction energy :

$$\partial U/\partial t = 0 \tag{56.19}$$

with

$$U = \sum_a n_a \int d\boldsymbol{p}\, \frac{p^2}{2m_a} f_a + \tfrac{1}{2} \sum_a \sum_b n_a n_b \int d\boldsymbol{r}\, d\boldsymbol{p}\, d\boldsymbol{p}'\, \Phi_{ab}(\boldsymbol{r}) \cdot f_a(\tilde{\boldsymbol{P}}(-\infty), t)\, f_b(\tilde{\boldsymbol{P}}'(-\infty), t) \,. \tag{56.20}$$

Thus, the contribution of the interactions to the internal energy (to first order in the retardation) is determined by the function (56.13).

In the local equilibrium state we have the following conservation law:

$$\frac{\tilde{\boldsymbol{P}}^2(-\infty)}{2m_a} + \frac{\tilde{\boldsymbol{P}}'^2(-\infty)}{2m_b} = \frac{p^2}{2m_a} + \frac{p'^2}{2m_b} + \tilde{\Phi}_{ab}(\boldsymbol{r}) \,, \tag{56.21}$$

where

$$\tilde{\Phi}_{ab}(\boldsymbol{r}) = \frac{e_a e_b}{r} e^{-r/r_\mathbf{D}} \,. \tag{56.22}$$

We then obtain from (56.13) the pair distribution function :

$$f_{ab}(\boldsymbol{r}) = e^{-\tilde{\Phi}_{ab}/k_\mathbf{B}T} = \exp\left[-\frac{e_a e_b}{k_\mathbf{B} T r} e^{-r/r_\mathbf{D}}\right] . \tag{56.23}$$

At short distances, this expression reduces to the Boltzmann distribution for particles interacting through Coulomb forces, whereas at large distances it corresponds to the Debye distribution.

From (56.20) , (56.23) we find the expression for the internal energy in local equilibrium :

$$U = \sum_a n_a \tfrac{3}{2} k_\mathbf{B}T + \tfrac{1}{2} \sum_a \sum_b n_a n_b \int d\boldsymbol{r}\, \frac{e_a e_b}{r} \exp\left[-\frac{e_a e_b}{k_\mathbf{B}T} \frac{e^{-r/r_\mathbf{D}}}{r}\right] . \tag{56.24}$$

To sum up, Eqs (56.14) – (56.16) define the collision integral for nonideal plasmas, combining the pair collision and the polarization approximations. We have seen how the effective potential, averaged over the dynamical polarization, enters the expressions of the thermodynamic functions. This potential also influences the transport coefficients, as we now show for the case of the electrical conductivity.

For frequencies ω of the external field much smaller than ω_L, the conductivity is defined by Eq. (48.19). If the conductivity is calculated on the basis of the Landau equation, the collision frequency in (48.19) is proportional to the Coulomb logarithm L (see (48.20)). Taking into account the averaged dynamical polarization, this factor must be changed into $L' = \ln\left(1 + r_\mathbf{D}^2 k_{\mathbf{max}}^2\right)^{\frac{1}{2}} > L$. Thus, the conductivity becomes smaller. For small values of the plasma parameter (i.e., for weakly nonideal plasmas), the corresponding decrease in conductivity is of order μ .

Equation (48.19) is derived on the basis of the collision integral I_{a1} alone (see Eq. (56.14)). The second term, I_{a2}, in that equation describes the effect of the nonideality in a spatially homogeneous plasma in the absences of an external field. It takes account of the retardation of the function f_a. In presence of an external field this is insufficient. For a complete account of the nonideality we must include the effect of the external field on the spectral function. This leads to a field-dependent collision integral I_a (see chapt. 8). For weak fields this amounts to adding to the collision integral (56.14) a corrective term proportional to the field. As a result, in the calculation of the conductivity the field $\boldsymbol{E}$ must be replaced by an effective field $\boldsymbol{E}_{\mathbf{eff}}$, and the conductivity itself is changed as follows :

$$\sigma \longrightarrow \sigma\left(1 - \frac{1}{6}\,\frac{e^2}{r_{\mathrm{D}} k_{\mathrm{B}} T}\right) \sim \sigma\,(1-\mu)\,.$$

We see that the field dependence of the collision integral leads to an additional decrease of the conductivity. The corresponding change is again proportional to the plasma parameter μ.

Thus, in the calculation of the conductivity, the nonideality of the plasma enters both through the effective potential and through the field dependence of the collision integral. In a spatially inhomogeneous plasma there is still another term in the collision integral (see chapt. 10). It yields corresponding corrections to the coefficients of diffusion, viscosity and heat conduction. All these terms are proportional to μ for weakly nonideal plasmas.

We calculated here the conductivity without consideration of the electron-electron collisions, which become important at low frequencies ($\omega \ll \nu_{ee}$) [42]. This, however, does not modify our conclusions about the decrease of the electrical conductivity as a result of the plasma nonideality.

57. INTERACTION OF CHARGED PARTICLES WITH PLASMONS

In the collision integrals for ideal and nonideal plasmas we took account of the fluctuations of δN_a, $\delta \boldsymbol{E}$ whose correlation time τ_{cor} is shorter than the mean free time, i.e., than the characteristic time of the function f_a. The fluctuations satisfying this condition can in turn be separated into two groups: those for which the correlation time is shorter than r_{D}/v_T, and those for which $r_{\mathrm{D}}/v_T < \tau_{\mathrm{cor}} < \tau_a$. The corresponding correlation radius for the first group is smaller than r_{D}, whereas for the second group :

$$r_{\mathrm{D}} < r_{\mathrm{cor}} < l_a\,. \tag{57.1}$$

We recall (see sect. 33) that the physically infinitesimal volume introduced for the derivation of the kinetic equation of plasmas has a characteristic length $l_{ph} \sim r_{\mathrm{D}}$. Hence, the fluctuations satisfying (57.1) cannot be completely

represented in terms of f_a, δE^{source} in (54.10). As a result, the collision integral (54.28) cannot be entirely expressed in terms of f_a. For the fluctuations of the group defined by (57.1) we need an additional equation.

The fluctuations with r_{cor} in the range (57.1) can be considered as random waves in a collisionless plasma. It is convenient to call them plasmons.

According to (32.13) and (32.14), for waves in a collisionless Coulomb plasma we have (we delete the prime of ω', and the $\parallel$-sign of $\varepsilon^{\parallel}$) :

$$\operatorname{Re} \varepsilon(\omega, \boldsymbol{k}) = 0 , \quad \gamma(\boldsymbol{k}) = \operatorname{Im} \varepsilon / (\partial \operatorname{Re} \varepsilon / \partial \omega) . \tag{57.2}$$

The first equation provides a link between ω and $\boldsymbol{k}$ (dispersion equation), and the second defines the damping rate. These equations are valid if :

$$\omega(\boldsymbol{k}) > \gamma(\boldsymbol{k}) > \nu_a . \tag{57.3}$$

The imaginary part of the dielectric constant is

$$\operatorname{Im} \varepsilon = - \sum_a \frac{4\pi^2 e_a^2 n_a}{k^2} \int d\boldsymbol{p} \; \delta(\omega - \boldsymbol{k}\cdot\boldsymbol{v}) \; \boldsymbol{k} \cdot \frac{\partial f_a}{\partial \boldsymbol{p}} \tag{57.4}$$

hence it is proportional (for $\boldsymbol{k} \parallel x$) to the derivative of $f_a(p_x)$ at $v_x = \omega/k$. For $\gamma(\boldsymbol{k}) \ll \omega(\boldsymbol{k})$ the phase velocity is much larger than the thermal speed. It follows that, for any distribution not too different from a Maxwellian, the number of particles having velocities near $v_x = \omega/k$ is small. This accounts for the smallness of $\operatorname{Im} \varepsilon$.

We now isolate in the collision integrals (54.24) , (54.26) for nonideal plasmas the contribution of the plasmons. In the range (57.1) we may make the following replacement in Eq. (54.25) :

$$\left(\frac{\partial \, \delta \boldsymbol{E}}{\partial t} \cdot \delta \boldsymbol{E} \right)_{\omega, \boldsymbol{k}, t} \rightarrow \frac{1}{2} \frac{\partial}{\partial t} \left(\delta \boldsymbol{E} \cdot \delta \boldsymbol{E} \right)_{\omega, \boldsymbol{k}, t} . \tag{57.5}$$

The neglected term is proportional to

$$\delta(\omega - \boldsymbol{k}\cdot\boldsymbol{v}) \; \boldsymbol{k} \cdot \frac{\partial f_a}{\partial \boldsymbol{p}} \frac{\partial}{\partial t} .$$

This term is of second order. Thus, the contribution of the plasmons to I_a^{ind} is defined by

$$\left(I_a^{\text{ind}} \right)_{pl} = \frac{e_a^2}{(2\pi)^4} \frac{\partial}{\partial \boldsymbol{p}} \cdot \operatorname{Re} \int_0^{\infty} d\tau \int d\omega \, d\boldsymbol{k} \, \frac{\boldsymbol{k}}{k^2} \exp\left[-\Delta\tau + i(\omega - \boldsymbol{k}\cdot\boldsymbol{v})\tau \right]$$
$$\cdot \left(\delta \boldsymbol{E} \cdot \delta \boldsymbol{E} \right)_{\omega, \boldsymbol{k}, t-\tau/2} \; \boldsymbol{k} \cdot \frac{\partial}{\partial \boldsymbol{p}} f_a(\boldsymbol{p}, t-\tau) . \tag{57.6}$$

It is understood that in this expression we must only keep terms of the first order in the retardation.

To the same approximation we now write the contribution of the plasmons to

I_a^{source} by going back to Eq. (54.26). Because of the conditions (57.2), (57.3), the terms of this equation involving time derivatives are of second order; hence

$$\left(I_a^{\text{source}}\right)_{pl} = -\frac{e_a}{n_a}\frac{1}{(2\pi)^4}\frac{\partial}{\partial \boldsymbol{p}}\cdot\int d\omega\, d\boldsymbol{k}\ \text{Re}\left[\frac{1}{\varepsilon^*}\left(\delta N_a\, \delta E\right)^{\text{source}}_{\omega,\boldsymbol{k},\boldsymbol{p}}\right]. \quad (57.7)$$

Using Eq. (52.12), we find from Eqs (57.6), (57.7) the following expression for the contribution of the particle-plasmon interactions to the collision integral of a nonideal plasma [7,8]:

$$\left(I_a\right)_{pl} = \frac{e_a^2}{(2\pi)^4}\frac{\partial}{\partial \boldsymbol{p}}\cdot \text{Re}\int_0^\infty d\tau \int d\omega\, d\boldsymbol{k}\ \frac{\boldsymbol{k}}{k^2}\exp\left[-\Delta\tau + i(\omega - \boldsymbol{k}\cdot\boldsymbol{v})\tau\right]$$

$$\left\{\left(\delta E\cdot\delta E\right)_{\omega,\boldsymbol{k},t-\tau/2}\ \boldsymbol{k}\cdot\frac{\partial f_a(\boldsymbol{p},t-\tau)}{\partial \boldsymbol{p}} + \frac{8\pi\ \text{Im}\,\varepsilon(\omega,\boldsymbol{k})}{|\varepsilon(\omega,\boldsymbol{k})|^2}\, f_a(\boldsymbol{p},t-\tau)\right\}. \quad (57.8)$$

In this expression, only terms of first order in the retardation are significant. In equilibrium the right-hand side of (57.8) vanishes. This is readily verified by substituting for f_a a Maxwellian, and for the spectral density of the field fluctuations the expression (36.4).

Equation (57.8) can be simplified by noting that for small $\text{Im}\,\varepsilon$:

$$\frac{\text{Im}\,\varepsilon}{|\varepsilon|^2} \to \pi\,(\text{sign}\,\omega)\ \delta\left[\text{Re}\,\varepsilon(\omega,\boldsymbol{k})\right]. \quad (57.9)$$

Equation (57.8) must be completed by adding an equation for $(\delta E\cdot\delta E)_{\omega,\boldsymbol{k}}$, defining the spectral distribution of the plasmon energy. We start from the balance equation (54.11). The expression of $\text{Re}\,(\delta \boldsymbol{j}\cdot\delta E)_{\omega,\boldsymbol{k},t}$ is obtained from (54.16):

$$\text{Re}\left(\delta \boldsymbol{j}\cdot\delta E\right)_{\omega,\boldsymbol{k},t} = \text{Re}\left(\delta \boldsymbol{j}^{\text{ source}}\cdot\delta E\right)_{\omega,\boldsymbol{k},t}$$

$$-\ \text{Re}\sum_a \frac{e_a n_a}{k^2}\int_0^\infty d\tau \int d\boldsymbol{p}\ \boldsymbol{k}\cdot\boldsymbol{v}\exp\left[-\Delta\tau + i(\omega - \boldsymbol{k}\cdot\boldsymbol{v})\tau\right]$$

$$\cdot\left(\delta E\cdot\delta E\right)_{\omega,\boldsymbol{k},t-\tau/2}\ \boldsymbol{k}\cdot\frac{\partial f_a(\boldsymbol{p},t-\tau)}{\partial \boldsymbol{p}} \quad (57.10)$$

where the simplification (57.5) was used.

Consider the first term on the right-hand side. We substitute (54.10) for δE^* and retain only first-order terms:

$$\text{Re}\left(\delta \boldsymbol{j}^{\text{source}}\cdot\delta E\right)_{\omega,\boldsymbol{k},t} = \text{Re}\,\frac{1}{\varepsilon^*}\left(\delta \boldsymbol{j}\cdot\delta E\right)^{\text{source}}_{\omega,\boldsymbol{k},t}. \quad (57.11)$$

Hence, using (52.12) we find

$$\mathrm{Re}\left(\delta f^{\mathrm{source}}\cdot\delta E\right)_{\omega',k,t} = -\frac{\mathrm{Im}\,\varepsilon}{4\pi}\cdot\frac{\left(\delta E\cdot\delta E\right)^{\mathrm{source}}_{\omega,k,t}}{|\varepsilon(\omega,k)|^2}\,. \tag{57.12}$$

In the second term on the right-hand side of (57.10), we expand in powers of τ and retain zeroth and first order terms. Using (54.9) and the expressions

$$\mathrm{Re}\,\varepsilon(\omega,k) = 1+\sum_a \frac{4\pi e_a^2 n_a}{k^2}\,P\int dp\,\frac{k\cdot(\partial f_a/\partial p)}{\omega-k\cdot v}$$

$$\mathrm{Im}\,\varepsilon(\omega,k) = -\sum_a \frac{4\pi^2 e_a^2 n_a}{k^2}\int dp\,\delta(\omega-k\cdot v)\,k\cdot\frac{\partial f_a}{\partial p} \tag{57.13}$$

which follows from (35.5), we express this term in the form

$$\mathrm{Im}\,\omega\varepsilon\,\frac{\left(\delta E\cdot\delta E\right)_{\omega,k,t}}{4\pi}-\frac{\partial\omega(\mathrm{Re}\,\varepsilon-1)}{\partial\omega}\cdot\frac{\partial}{\partial t}\,\frac{\left(\delta E\cdot\delta E\right)_{\omega,k,t}}{8\pi}$$
$$+\frac{\partial^2\omega\,\mathrm{Re}\,\varepsilon}{\partial t\,\partial\omega}\,\frac{\left(\delta E\cdot\delta E\right)_{\omega,k,t}}{4\pi}\,. \tag{57.14}$$

From (57.2) , (57.12) and (57.14), we obtain the following equation for the spectral density of the plasmons :

$$\frac{\partial}{\partial t}\left(\delta E\cdot\delta E\right)_{\omega,k,t} = -2\gamma(k)\left[\left(\delta E\cdot\delta E\right)_{\omega,k,t}-\frac{\left(\delta E\cdot\delta E\right)^{\mathrm{source}}_{\omega,k,t}}{|\varepsilon(\omega,k)|^2}\right]$$
$$-2\,\frac{\partial^2\,\mathrm{Re}\,\varepsilon}{\partial t\,\partial\omega}\left[\frac{\partial\,\mathrm{Re}\,\varepsilon}{\partial\omega}\right]^{-1}\left(\delta E\cdot\delta E\right)_{\omega,k,t}\,. \tag{57.15}$$

If in deriving this equation we take into account the time-dependence of the frequency, due to the time-dependence of the function f_a involved in the dispersion equation (57.2), we get on the right-hand side of (57.15) an additional term :

$$-\frac{\partial\omega}{\partial t}\,\frac{\partial^2\,\mathrm{Re}\,\varepsilon}{\partial\omega^2}\left[\frac{\partial\,\mathrm{Re}\,\varepsilon}{\partial\omega}\right]^{-1}\left(\delta E\cdot\delta E\right)_{\omega,k,t}\,. \tag{57.16}$$

A corresponding additional term appears also in the collision integral $(I_a)_{pl}$.

It follows from (57.15) and (57.16) that the effective damping rate of the spectral density is given by the expression :

$$\gamma_{\mathbf{eff}} = \gamma(k)+\frac{\partial}{\partial t}\,\ln\frac{\partial\,\mathrm{Re}\,\varepsilon}{\partial\omega}+\frac{1}{2}\,\frac{\partial\omega}{\partial t}\,\frac{\partial^2\,\mathrm{Re}\,\varepsilon}{\partial\omega^2}\left[\frac{\partial\,\mathrm{Re}\,\varepsilon}{\partial\omega}\right]^{-1} \tag{57.17}$$

The kinetic equation with $(I_a)_{pl}$ satisfies the conservation law of the energy, with due account of the plasmon energy.

We recall that Eqs (57.8) and (57.15) only describe the interactions of the particles with those plasmons for which the damping rate is larger than the collision frequency, but smaller than the plasma frequency.

The separation of the plasmon contribution to the kinetic equations only makes sense for states very far from equilibrium. Near equilibrium, the contribution of the plasmons is small [43 , 44]. Such non-equilibrium states appear in the presence of external fields or of other external actions, which lead to the onset and growth of instabilities in the plasma.

We consider as an example the instability produced by the passage through the plasma of a beam of fast electrons ('beam instability'). Let u be the average velocity of the beam electrons, with $u \gg v_T$ (where v_T is the thermal velocity of the plasma electrons). It is known (see, e.g., [15 , 28]) that for $n_1 \ll n$ (n_1 being the number density in the beam) the damping rate is largest for waves with $\omega = \omega_L$ and $k = \omega_L/u \ll \omega_L/v_T \sim 1/r_{\mathbf{D}}$. Let us estimate this damping rate, by using (57.2). If f_e is Maxwellian, then

$$\gamma = \left(\frac{\pi}{8}\right)^{\frac{1}{2}} \omega_L \frac{1}{(r_{\mathbf{D}} k)^3} \exp\left(-\frac{1}{2 r_{\mathbf{D}}^2 k^2}\right). \qquad (57.18)$$

This is simply the Landau damping. Substituting $k = \omega_L/u$ and noting that $\omega_L r_{\mathbf{D}} = v_T/(2)^{\frac{1}{2}}$, we get

$$\gamma = \pi^{\frac{1}{2}} (u/v_T)^3 \exp\left(-\frac{u^2}{v_T^2}\right). \qquad (57.19)$$

We see that γ decreases exponentially with increasing u/v_T. Thus for $u \gg v_T$ the damping rate calculated from (57.19) becomes smaller than the collision frequency and the condition (57.3) is violated. This implies that for the calculation of the spectral densities of δN_a, $\delta \boldsymbol{E}$ it is insufficient to consider only the range of frequencies and wavenumbers corresponding to the collisionless approximation. We must specifically take into account the collisions which, in turn, involve the fluctuations in the domain $k > 1/r_{\mathbf{D}}$.

Let us consider another example. In a non-isothermal plasma there exists the possibility of the so-called ion-acoustic instability. The latter sets in whenever the relative velocity of electrons and ions exceeds the ion sound speed $v_s = (k_{\mathbf{B}} T_e/m_i)^{\frac{1}{2}}$. From the formula (32.27) for the damping rate of the ion-sound waves follows that $\gamma(\boldsymbol{k})$ has a maximum value for $k r_{\mathbf{D}e} = 1$. This value is :

$$\gamma_{\mathbf{max}} = \left(\frac{\pi}{8} \frac{m_e}{m_i}\right)^{\frac{1}{2}} v_s k_{\max} \approx \omega_e \frac{m_e}{m_i} . \qquad (57.20)$$

On the other hand, the electron-electron collision frequency is found from (38.9) :

$$\nu_{ee} \sim \omega_e \mu \qquad (57.21)$$

where μ is the plasma parameter. Hence, whenever

$$m_e/m_i < \mu \tag{57.22}$$

the condition $\gamma(k) > \nu_a$ (see (57.3)) is again violated, and the calculation of the spectral density of the fluctuations in the plasmon region must take account of the collisions.

The consideration of these examples shows that there exist situations in which there are important contributions to the collision integral coming from long-range and long-living correlations. These are correlations for which $\tau_{\mathbf{cor}}$ is larger than the mean free time and $r_{\mathbf{cor}}$ exceeds the mean free path. The mean free time and path are defined by the collision integral in which only fluctuations with $\tau_{\mathbf{cor}} < 1/\nu_a$ and $r_{\mathbf{cor}} < l_a$ are involved.

From this discussion follows the necessity of constructing a kinetic theory for plasmas, taking into account the long-range and long-living fluctuations. This problem will be studied in Chapter 11.

CHAPTER 10

The Spatially Inhomogeneous Nonideal Plasma

58. RELATION BETWEEN COLLISION INTEGRAL AND SPECTRAL DENSITY OF FLUCTUATIONS

We use the following expression for the collision integral :

$$I_a(x,t) = -\frac{e_a}{n_a}\frac{\partial}{\partial \boldsymbol{p}} \cdot \left\langle \delta N_a \, \delta E \right\rangle_{x,\boldsymbol{r},t} . \tag{58.1}$$

For a Coulomb plasma, expressing δE in terms of δN_a through the Poisson equation we write the equation in the form

$$I_a(x,t) = \frac{e_a}{n_a}\sum_b e_b \frac{\partial}{\partial \boldsymbol{p}} \cdot \int dx' \, \frac{\partial}{\partial \boldsymbol{r}} \frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|} \left\langle \delta N_a \, \delta N_b \right\rangle_{x,x',t} \tag{58.2}$$

In the spatially homogeneous case, the function $\langle \delta N_a \, \delta N_b \rangle_{x,x',t}$ depends only on the difference of the co-ordinates $\boldsymbol{r}-\boldsymbol{r}'$. For inhomogeneous plasmas, we introduce, as in section 15, instead of $\boldsymbol{r},\boldsymbol{r}'$ the variables $\boldsymbol{r}-\boldsymbol{r}' = \boldsymbol{r}_{ab}$ and $(\boldsymbol{r}+\boldsymbol{r}')/2 = \boldsymbol{r}-\boldsymbol{r}_{ab}/2$ and perform an expansion in powers of $\boldsymbol{r}_{ab}\cdot\partial/\partial\boldsymbol{r}$. To first order, we have

$$\left\langle \delta N_a \, \delta N_b \right\rangle_{\boldsymbol{r},\boldsymbol{p},\boldsymbol{r}',\boldsymbol{p}',t} = \left(1-\tfrac{1}{2}\,\boldsymbol{r}_{ab}\cdot\frac{\partial}{\partial \boldsymbol{r}}\right)\left\langle \delta N_a \, \delta N_b \right\rangle_{\boldsymbol{r}_{ab},\boldsymbol{r},\boldsymbol{p},\boldsymbol{p}',t} . \tag{58.3}$$

This expression is analogous to (15.2). Substituting it into (58.2), we get

$$I_a(\boldsymbol{r},\boldsymbol{p},t) = \frac{e_a}{n_a}\sum_b e_b \frac{\partial}{\partial \boldsymbol{p}} \cdot \int d\boldsymbol{r}_{ab}\, d\boldsymbol{p}' \left(\frac{\partial}{\partial \boldsymbol{r}_{ab}} \frac{1}{r_{ab}}\right)$$

$$\cdot \left(1-\tfrac{1}{2}\,\boldsymbol{r}_{ab}\cdot\frac{\partial}{\partial \boldsymbol{r}}\right)\left\langle \delta N_a \, \delta N_b \right\rangle_{\boldsymbol{r}_{ab},\boldsymbol{r},\boldsymbol{p},\boldsymbol{p}',t} \tag{58.4}$$

We introduce Fourier transforms with respect to $\boldsymbol{r}_{ab}$ and use the formulae :

$$\frac{1}{r} = \frac{1}{(2\pi)^3}\int dk\ \frac{4\pi}{k^2}\ e^{i\boldsymbol{k}\cdot\boldsymbol{r}}$$

$$\frac{1}{2}\, r_i\, \frac{\partial}{\partial r_j}\ \frac{1}{r} = -\frac{1}{(2\pi)^3}\int d\boldsymbol{k}\ \frac{4\pi}{k^2}\left(\frac{1}{2}\,\delta_{ij} - \frac{k_i k_j}{k^2}\right) e^{i\boldsymbol{k}\cdot\boldsymbol{r}} . \tag{58.5}$$

Then

$$I_a = \frac{1}{(2\pi)^3 n_a}\sum_a e_a e_b \frac{\partial}{\partial p_i}\int d\boldsymbol{k}\ d\boldsymbol{p}'\ \mathrm{Re}\,\frac{4\pi}{k^2}\Big[-ik_i + \left(\frac{1}{2}\,\delta_{ij} - \frac{k_i k_j}{k^2}\right)\frac{\partial}{\partial r_j}\Big]\left(\delta N_a\ \delta N_b\right)_{\boldsymbol{k},\,\boldsymbol{r},\,\boldsymbol{p},\,\boldsymbol{p}',\,t} . \tag{58.6}$$

Using the relation

$$\left(\delta N_a\ \delta\boldsymbol{E}\right)_{\boldsymbol{k},\,\boldsymbol{p},\,t} = \frac{i\boldsymbol{k}}{k^2}\sum_b 4\pi e_b\int d\boldsymbol{p}'\left(\delta N_a\ \delta N_b\right)_{\boldsymbol{k},\,\boldsymbol{p},\,\boldsymbol{p}',\,t}$$

we express the collision integral (58.6) in terms of the spectral density of the fluctuations of $\delta\boldsymbol{K}$, δN_a :

$$I_a(\boldsymbol{r},\boldsymbol{p},t) = -\frac{e_a}{n_a}\,\frac{1}{(2\pi)^3}\,\frac{\partial}{\partial\boldsymbol{p}}\cdot\int d\boldsymbol{k}\ \mathrm{Re}\left(\delta N_a\ \delta\boldsymbol{E}\right)_{\boldsymbol{k},\,\boldsymbol{r},\,\boldsymbol{p},\,t}$$
$$+\ \frac{e_a}{n_a(2\pi)^3}\,\frac{\partial}{\partial p_i}\int d\boldsymbol{k}\left(\frac{\delta_{ij}}{2} - \frac{k_i k_j}{k^2}\right)\frac{1}{k^2}\,\frac{\partial}{\partial r_j}\ \mathrm{Im}\left(\delta N_a\ \boldsymbol{k}\cdot\delta\boldsymbol{E}\right)_{\boldsymbol{k},\,\boldsymbol{r},\,\boldsymbol{p},\,t} . \tag{58.7}$$

Comparing this with the homogeneous case, we see that the collision integral involves an extra term, related to the imaginary part of the spectral density $(\delta N_a\,\delta\boldsymbol{E})_{\boldsymbol{k},\boldsymbol{p},t}$ [8] .

In order to find the explicit form of the collision integral, we must express this spectral density in terms of the function f_a with due account of the retardation and of the spatial inhomogeneity, i.e., find a solution analogous to (15.4). This will be done in the next section. Here we only note that, as in section 15, the collision integral can be represented as a sum of three terms

$$I_a(r,p,t) = I_{a_1} + I_{a_2} + I_{a_3} . \tag{58.8}$$

In the next paragraph we look for explicit forms, based on the models investigated in sections 55, 56.

59. THE COLLISION INTEGRAL

In order to obtain the contribution of the interactions at both short and long distances we use Eqs (56.11) , (56.12), with account of the spatial inhomogeneity. They can be written in the following form, analogous to (2.11) , (2.18):

$$\left[\frac{\partial}{\partial t} + \boldsymbol{v}\cdot\frac{\partial}{\partial\boldsymbol{r}} + e_a\left(\boldsymbol{E}_0 + \boldsymbol{E}\right)\cdot\frac{\partial}{\partial\boldsymbol{p}}\right] f_a = \sum_b n_b\int dx'\ \frac{\partial\Phi_{ab}}{\partial\boldsymbol{r}}\cdot\frac{\partial}{\partial\boldsymbol{p}}\ g_{ab}(x,x',t) \equiv I_a \tag{59.1}$$

$$\left[\frac{\partial}{\partial t}+\boldsymbol{v}\cdot\frac{\partial}{\partial \boldsymbol{r}}+\boldsymbol{v}'\cdot\frac{\partial}{\partial \boldsymbol{r}'}-\frac{\partial\tilde{\Phi}_{ab}}{\partial \boldsymbol{r}}\cdot\frac{\partial}{\partial \boldsymbol{p}}-\frac{\partial\tilde{\Phi}_{ab}}{\partial \boldsymbol{r}'}\cdot\frac{\partial}{\partial \boldsymbol{p}'}\right]f_{ab}(x,x',t)$$
$$=\left(\frac{\partial}{\partial t}+\boldsymbol{v}\cdot\frac{\partial}{\partial \boldsymbol{r}}+\boldsymbol{v}'\cdot\frac{\partial}{\partial \boldsymbol{r}'}\right)f_a(x,t)\,f_b(x',t)\,. \tag{59.2}$$

We recall that the function $\langle \delta N_a\,\delta N_b\rangle$ appearing in (58.2) is related to the correlation function g_{ab} as follows :

$$\left\langle \delta N_a\,\delta N_b\right\rangle_{x,x',t}=n_a n_b g_{ab}+\delta_{ab}\,\delta(x-x')\,f_a n_a\,. \tag{59.3}$$

The term involving $\delta(x-x')$ does not contribute to the collision integral; hence the expression (58.2) and (59.1) are equivalent.

In (59.2), $\tilde{\Phi}_{ab}$ is the effective potential, taking into account the average dynamical polarization of the plasma. We use for it the expression (55.1). The action of the external field is neglected in (59.2): this is allowed for frequencies $\omega_0 \ll \omega_L$, if $eEr_{\mathrm{D}} \ll k_{\mathrm{B}}T$ (see sect. 46).

The solution of (59.2), to first order in $\boldsymbol{r}_{ab}\cdot(\partial/\partial\boldsymbol{r})$ and $\tau(\partial/\partial t)$ and assuming the complete weakening of the initial correlations, is analogous to (15.4):

$$f_{ab}(\boldsymbol{r}_{ab},\boldsymbol{r},\boldsymbol{p},\boldsymbol{p}',t)=f_{ab}(\boldsymbol{r},\tilde{\boldsymbol{P}}(-\infty),t)\,f_b(\boldsymbol{r},\tilde{\boldsymbol{P}}'(-\infty),t)$$
$$-\left(\frac{\partial}{\partial t}+\frac{\boldsymbol{v}+\boldsymbol{v}'}{2}\cdot\frac{\partial}{\partial\boldsymbol{r}}\right)\int_0^\infty d\tau\,\tau\,\frac{d}{d\tau}\,f_a(\boldsymbol{r},\tilde{\boldsymbol{P}}(-\tau),t)\,f_b(\boldsymbol{r},\tilde{\boldsymbol{P}}'(-\tau),t)\,. \tag{59.4}$$

Here $\tilde{\boldsymbol{P}}(-\tau),\tilde{\boldsymbol{P}}'(-\tau)$ are the initial momenta of two particles interacting through $\tilde{\Phi}_{ab}$. For the spatially homogeneous case (59.4) reduces to (56.13).

The collision integral involves the correlation function, which can be expressed in a form analogous to (58.3) :

$$g_{ab}(\boldsymbol{r},\boldsymbol{p},\boldsymbol{r}',\boldsymbol{p}',t)=\left(1-\tfrac{1}{2}\,\boldsymbol{r}_{ab}\cdot\frac{\partial}{\partial\boldsymbol{r}}\right)g_{ab}(\boldsymbol{r}_{ab},\boldsymbol{r},\boldsymbol{p},\boldsymbol{p}',t) \tag{59.5}$$

The function

$$g_{ab}(\boldsymbol{r}_{ab},\boldsymbol{p},\boldsymbol{r}',\boldsymbol{p}',t)=f_{ab}(\boldsymbol{r}_{ab},\boldsymbol{r},\boldsymbol{p},\boldsymbol{p}',t)$$
$$-f_a(\boldsymbol{r},\boldsymbol{p},t)\,f_b(\boldsymbol{r},\boldsymbol{p}',t) \tag{59.6}$$

is obtained from (59.4).

From the results (59.1), (59.4) follows that the collision integral can be expressed in the form (58.8) with the following expressions, analogous to Eqs (15.6)−(15.8) for gases:

$$I_{a_1}=\sum_b n_b\int d\boldsymbol{r}'\,d\boldsymbol{p}'\,\frac{\partial\Phi_{ab}}{\partial\boldsymbol{r}}\cdot\frac{\partial}{\partial\boldsymbol{p}}\,f_a(\boldsymbol{r},\tilde{\boldsymbol{P}}(-\infty),t)\,f_b(\boldsymbol{r},\tilde{\boldsymbol{P}}'(-\infty),t) \tag{59.7}$$

$$I_{a_2}=-\sum_b n_b\int_0^\infty d\tau\int d\boldsymbol{r}'\,d\boldsymbol{p}'\,\frac{\partial\Phi_{ab}}{\partial\boldsymbol{r}}\cdot\frac{\partial}{\partial\boldsymbol{p}}\left(\frac{\partial}{\partial t}+\frac{\boldsymbol{v}+\boldsymbol{v}'}{2}\cdot\frac{\partial}{\partial\boldsymbol{r}}\right)$$
$$\tau\,\frac{d}{d\tau}\,f_a(\boldsymbol{r},\tilde{\boldsymbol{P}}(-\tau),t)\,f_b(\boldsymbol{r},\tilde{\boldsymbol{P}}'(-\tau),t) \tag{59.8}$$

$$I_{a_3} = -\sum_b \frac{1}{2} n_b \int d\boldsymbol{r}' \, d\boldsymbol{p}' \, \frac{\partial \Phi_{ab}}{\partial \boldsymbol{r}} \cdot \frac{\partial}{\partial \boldsymbol{p}} \left(\boldsymbol{r}_{ab} \cdot \frac{\partial}{\partial \boldsymbol{r}} \right)$$
$$\cdot \left[f_a(\boldsymbol{r}, \tilde{\boldsymbol{P}}(-\infty), t) f_b(\boldsymbol{r}, \tilde{\boldsymbol{P}}'(-\infty), t) - f_a(\boldsymbol{r}, \boldsymbol{p}, t) f_b(\boldsymbol{r}, \boldsymbol{p}', t) \right]. \quad (59.9)$$

Let us compare these expressions with those for the homogeneous plasma. Equation (59.7) differs from (56.15) in the dependence of f_a, f_b on the position $\boldsymbol{r}$. (59.8) differs from (56.16) by the fact that the operator $\partial/\partial t$ is changed into $\partial/\partial t + \frac{1}{2}(\boldsymbol{v}+\boldsymbol{v}') \cdot (\partial/\partial \boldsymbol{r})$. The term I_{a_3} vanishes in the homogeneous case. As in (15.8), the second bracketed term in (59.9) compensates an analogous term coming from the average field $\boldsymbol{E}$ on the left-hand side of (59.1), as will be seen below.

60. PROPERTIES OF THE COLLISION INTEGRAL

For the collision integral (58.8), (59.7)–(59.9) the property (56.17) holds only for $\phi_a = 1$. For $\phi_a = \boldsymbol{p}$, (56.17) holds only for the first two terms of the collision integral; for I_{a_3} we find

$$\sum_a n_a \int d\boldsymbol{p} \; p_i I_{a_3} = -\sum_a \frac{\partial}{\partial r_j} \Delta P^a_{ij} \quad (60.1)$$

where ΔP^a_{ij} is the contribution of the interactions to the tensor P^a_{ij}. It is defined by

$$\Delta P^a_{ij} = -\sum_b \frac{n_a n_b}{2} \int d\boldsymbol{r}_{ab} \, d\boldsymbol{p} \, d\boldsymbol{p}' \, \frac{(r_{ab} r_{ab})_{ij}}{r_{ab}} \cdot \frac{\partial \Phi_{ab}}{\partial r_{ab}}$$
$$\cdot \; f_a(\boldsymbol{r}, \tilde{\boldsymbol{P}}(-\infty), t) \; f_b(\boldsymbol{r}, \tilde{\boldsymbol{P}}'(-\infty), t) \quad (60.2)$$

which is analogous to (15.13) for gases. The corresponding contribution to the pressure is:

$$\Delta p_a = \frac{1}{3} \Delta P^a_{ii} \; . \quad (60.3)$$

In the local equilibrium approximation (for $T_a = T$) we find :

$$\Delta p_a = -\sum_b \frac{2}{3} \pi n_a n_b \int_0^\infty dr \, r^3 \, \Phi'_{ab} \, e^{-\tilde{\Phi}_{ab}/k_B T} \quad (60.4)$$

where

$$\tilde{\Phi}_{ab} = e_a e_b \left(e^{-r/r_D} / r \right) .$$

In the Debye approximation we find, recalling that $\sum e_a n_a = 0$:

$$\Delta p_a = \sum_b \frac{2\pi n_a n_b}{3 k_B T} \int_0^\infty dr \, r^3 \, \Phi'_{ab} \, \tilde{\Phi}_{ab} = -\frac{e_a^2 n_a}{6 r_D} . \quad (60.5)$$

Hence, the contribution of the interactions to the total pressure is :

$$\Delta p = \sum_a \Delta p_a = -\sum_a \frac{e_a^2 n_a}{6 r_\mathrm{D}} = -\frac{1}{3}\left(\frac{\pi}{k_\mathrm{B}T}\right)^{\frac{1}{2}}\left(\sum_a e_a^2 n_a\right)^{\frac{3}{2}} \tag{60.6}$$

which is precisely the well-known expression from the Debye theory (see Eq. (75.15) in ref. [45]).

For $\phi_a = p^2/2m_a$ Eq. (56.17) holds only for I_{a_1}. For I_{a_2} we get :

$$\sum_a n_a \int d\boldsymbol{p}\, \frac{p^2}{2m_a} I_{a_2} = -\sum_a\sum_b \tfrac{1}{2}\, n_a n_b \int d\boldsymbol{r}_{ab}\, d\boldsymbol{p}\, d\boldsymbol{p}'\, \Phi_{ab}\left(\frac{\partial}{\partial t} + \frac{\boldsymbol{v}+\boldsymbol{v}'}{2}\cdot\frac{\partial}{\partial \boldsymbol{r}}\right)$$

$$\cdot\, f_a(\boldsymbol{r}, \tilde{\boldsymbol{P}}(-\infty), t)\, f_b(\boldsymbol{r}, \tilde{\boldsymbol{P}}'(-\infty), t) \tag{60.7}$$

which is analogous to (15.21). Finally, I_{a_3} yields

$$\sum_a n_a \int d\boldsymbol{p}\, \frac{p^2}{2m_a} I_{a_3} = \sum_a\sum_b \tfrac{1}{4} n_a n_b \frac{\partial}{\partial r_j}\int d\boldsymbol{r}_{ab}\, d\boldsymbol{p}\, d\boldsymbol{p}'(\boldsymbol{v}+\boldsymbol{v}')_i$$

$$\frac{(\boldsymbol{r}_{ab}\boldsymbol{r}_{ab})_{ij}}{2}\, \Phi'_{ab}(\boldsymbol{r}_{ab})\, f_a(\boldsymbol{r}, \tilde{\boldsymbol{P}}(-\infty), t)\, f_b(\boldsymbol{r}, \tilde{\boldsymbol{P}}'(-\infty), t) \tag{60.8}$$

by analogy with (15.25).

In the local equilibrium approximation we find from (60.7), (60.8) :

$$\sum_a n_a \int d\boldsymbol{p}\, \frac{p^2}{2m_a}\left(I_{a_2}+I_{a_3}\right) = -\sum_a\left[\frac{\partial \Delta U_a}{\partial t} + \frac{\partial}{\partial \boldsymbol{r}}\cdot \boldsymbol{u}\left(\Delta U_a + \Delta p_a\right)\right] \tag{60.9}$$

where

$$\Delta U_a = \sum_b 2\pi n_a n_b \int dr\, r^2\, \Phi_{ab}\, e^{-\tilde{\Phi}_{ab}/k_\mathrm{B}T} \tag{60.10}$$

is the contribution of the correlations to the internal energy density of the particles of species a. In the Debye approximation, (60.10) reduces to

$$\Delta U_a = -\sum_b \frac{2\pi n_a n_b}{k_\mathrm{B}T}\int_0^\infty dr\, r^2\, \Phi_{ab}\, \tilde{\Phi}_{ab} = -\frac{e_a^2 n_a}{2 r_\mathrm{D}}. \tag{60.11}$$

The contribution of the interactions to the total energy density is

$$\Delta U = \sum_a \Delta U_a = -\sum_a \frac{e_a^2 n_a}{2 r_\mathrm{D}} = -\left(\frac{\pi}{k_\mathrm{B}T}\right)^{\frac{1}{2}}\left(\sum_a e_a^2 n_a\right)^{\frac{3}{2}} \tag{60.12}$$

which is the correct expression from the Debye theory (Eq. (75.11) from ref. [45]).

Hence, from the kinetic equations (58.8), (59.7) – (59.9) we obtain the hydrodynamical equations of the nonideal plasma, taking into account the contribution of the interactions to the thermodynamic functions.

In the kinetic equation (59.1), the contribution of the interactions is divided into two parts. One of them is defined through the correlation function g_{ab} and is contained in the collision integral. The other involves the functions

f_a, f_b and is included in the average field term:

$$e_a \boldsymbol{E} \cdot \frac{\partial f_a}{\partial \boldsymbol{p}} = -\sum_b n_b \int dx' \frac{\partial \Phi_{ab}}{\partial \boldsymbol{r}} \cdot \frac{\partial}{\partial \boldsymbol{p}} f_a(x,t)\, f_b(x',t) \,. \tag{60.13}$$

Such a division of the contribution of the interactions is very convenient in many cases. It is, however, not absolute, as it depends on the relation between the time scales of the field and of the distribution function. This relation, in turn, depends on the 'distance' of the state from equilibrium.

Indeed, in the hydrodynamical regime the change of the distribution function is determined by the change of the hydrodynamical functions $n_a, \boldsymbol{u}_a, T_a$ which is characterized by the hydrodynamical time τ_a^h:

$$\tau_a^h \sim \tau_a \frac{L^2}{l_a^2} \gg \tau_a \,, \quad \text{as} \quad L \gg l_a \tag{60.14}$$

Here L is a characteristic length of the hydrodynamical process, and τ_a and l_a are, respectively, the mean free time and the mean free path of the particles of species a. The hydrodynamical regime is valid when $L \gg l_a$. Thus if in the kinetic description the average electric field has a relaxation time longer than τ_a, then in the hydrodynamic description the average field has a relaxation time longer than $\tau_a^h \gg \tau_a$. As a result, as the system approaches equilibrium, the collision integral must include fluctuations with increasingly longer correlation times. The part of the interactions attributed to (60.13) therefore changes. These questions will be discussed in more detail in the next chapter.

Before concluding this section, we note that by using the results of sections 26, 58, the contributions to the thermodynamic functions due to the correlations, can be expressed in terms of correlations of $\delta N_a, \delta \boldsymbol{E}$. For instance, from Eq. (58.4) follows that the collision integral I_{a_3} can be written as

$$I_{a_3} = -\frac{e_a}{2n_a} \sum_b e_b \frac{\partial}{\partial \boldsymbol{p}} \cdot \int d\boldsymbol{r}_{ab}\, d\boldsymbol{p}' \left(\frac{\partial}{\partial \boldsymbol{r}_{ab}} \frac{1}{r_{ab}}\right) \left(\boldsymbol{r}_{ab} \cdot \frac{\partial}{\partial \boldsymbol{r}}\right) \cdot \left\langle \delta N_a\, \delta N_b \right\rangle_{\boldsymbol{r}_{ab}, \boldsymbol{r}, \boldsymbol{p}, \boldsymbol{p}', t} \,. \tag{60.15}$$

Hence the correlational part of the pressure tensor is:

$$\sum_a \Delta P_{ij}^a = -\sum_a \sum_b \frac{1}{2} \int d\boldsymbol{r}_{ab} \frac{(\boldsymbol{r}_{ab}\, \boldsymbol{r}_{ab})_{ij}}{r_{ab}} \Phi'_{ab} \left\langle \delta n_a\, \delta n_b \right\rangle_{\boldsymbol{r}_{ab}, \boldsymbol{r}, t} \tag{60.16}$$

where

$$\left\langle \delta n_a\, \delta n_b \right\rangle_{\boldsymbol{r}_{ab}, \boldsymbol{r}, t} = \delta_{ab}\, \delta(\boldsymbol{r}_{ab})\, n_a(\boldsymbol{r}, t) + n_a n_b\, g_{ab}(\boldsymbol{r}_{ab}, \boldsymbol{r}, t) \tag{60.17}$$

is the density correlation function of particles of species a, b. From (59.4) we obtain

$$\left\langle \delta n_a \, \delta n_b \right\rangle_{\boldsymbol{r}_{ab}, \boldsymbol{r}, t} = \delta_{ab} \, \delta(\boldsymbol{r}_{ab}) \, n_a(\boldsymbol{r}, t) + n_a n_b \int d\boldsymbol{p} \, d\boldsymbol{p}' \Big[f_a(\boldsymbol{r}, \tilde{\boldsymbol{P}}(-\infty), t) \, f_b(\boldsymbol{r}, \tilde{\boldsymbol{P}}'(-\infty), t) - f_a(\boldsymbol{r}, \boldsymbol{p}, t) \, f_b(\boldsymbol{r}, \boldsymbol{p}', t) \Big] \, . \tag{60.18}$$

In the local equilibrium state we have

$$\left\langle \delta n_a \, \delta n_b \right\rangle_{\boldsymbol{r}_{ab}, \boldsymbol{r}, t} = \delta_{ab} \, \delta(\boldsymbol{r}_{ab}) \, n_a(\boldsymbol{r}, t) + n_a n_b \left(e^{-\tilde{\Phi}_{ab}/k_{\mathbf{B}} T} - 1 \right) . \tag{60.19}$$

The contribution of the correlations to the internal energy density is :

$$\Delta U(\boldsymbol{r}, t) = \frac{1}{2} \sum_a \sum_b \int d\boldsymbol{r}_{ab} \, \Phi_{ab} \left\langle \delta n_a \, \delta n_b \right\rangle_{\boldsymbol{r}_{ab}, \boldsymbol{r}, t} . \tag{60.20}$$

It differs from the expression (60.7) by the inclusion of the self-interactions.

CHAPTER 11

Kinetic Theory of Fluctuations in a Plasma

61. EQUATIONS FOR THE SMOOTHED PHASE-SPACE DENSITY AND FIELD

In the kinetic theory of fluctuations in gases (chapter 4), we used two methods. One was based on the hierarchy for the smoothed distribution functions $f_1, f_2, \ldots$ The latter differs from the Bogolyubov hierarchy in that it takes the dissipation due to binary collisions into account from the very beginning. The second method was based on the equation (22.1) for the smoothed phase-space density in the six-dimensional position and momentum space.

We saw that the two methods were equivalent, but the phase-density method leads to a much simpler calculation of the long-range fluctuations. For this reason, it will be used for the construction of the theory of fluctuations in plasmas.

In the kinetic theory of fluctuations we consider long-range and long-living fluctuations, i.e. those for which the correlation length and time exceed, respectively, l_{ph}, τ_{ph}, the physically infinitesimal elements of kinetic theory. For a plasma, $l_{ph} \sim r_{\mathrm{D}}, \tau_{ph} \sim 1/\omega_L$ (see sect. 33), hence we consider here fluctuations for which

$$\tau_{\mathrm{cor}} > \omega_L^{-1} , \quad r_{\mathrm{cor}} > r_{\mathrm{D}} .$$

The short-range fluctuations, which satisfy the opposite inequalities, define the collision integral.

In the kinetic theory of gases Eq. (22.1) was derived on the basis of the Boltzmann equation, and reduces to the latter upon averaging in the first moment approximation (i.e., on neglect of the correlations). For plasmas one can choose to derive the equation on the basis of several possible kinetic equations: Landau, Balescu-Lenard or Boltzmann with account of the polarization.

We shall choose to use here the Boltzmann equation (56.11) with the collision integral (56.15) or, if the retardation and the inhomogeneity are accounted for, (58.8), (59.7) – (59.9). In these collision integrals both the strong

interactions at short distance and the long-range polarization effects are included.

We now write the equation for the smoothed phase-density $\tilde{N}_a(x,t)$. Hereafter, we shall omit writing the tilda, whenever this leads to no confusion. We first consider an ideal plasma and generalize the result later. For the former case we have

$$\left(\frac{\partial}{\partial t} + \boldsymbol{v}\cdot\frac{\partial}{\partial \boldsymbol{r}} + \boldsymbol{F}_{a0}\cdot\frac{\partial}{\partial \boldsymbol{p}}\right) N_a(x,t)$$

$$= \int dx' \sum_b \frac{\partial \Phi_{ab}}{\partial \boldsymbol{r}}\cdot\frac{\partial}{\partial \boldsymbol{p}}\left[N_a(x,t)\,N_b(x',t)\right.$$

$$\left. + N_a(\boldsymbol{r},\tilde{\boldsymbol{P}}(-\infty),t)\;N_b(\boldsymbol{r},\tilde{\boldsymbol{P}}'(-\infty),t)\right]. \tag{61.2}$$

Here, as in (56.13), $\tilde{\boldsymbol{P}}(-\tau)$, $\tilde{\boldsymbol{P}}'(-\tau)$ are the initial momenta (at time $t-\tau$) of the two particles interacting through the potential $\tilde{\Phi}_{ab}$, and Φ_{ab} is the Coulomb potential.

The second term on the right-hand side describes both the strong binary collisions at short distances and the collective interactions through the effective potential.

For nonideal plasmas in the first approximation in the retardation and the inhomogeneity, the right-hand side of (61.2) is replaced by:

$$\sum_b \int dx'\;\frac{\partial \Phi_{ab}}{\partial \boldsymbol{r}}\cdot\frac{\partial}{\partial \boldsymbol{p}}\left\{\left(1-\tfrac{1}{2}\boldsymbol{r}_{ab}\cdot\frac{\partial}{\partial \boldsymbol{r}}\right)\left[N_a(\boldsymbol{r},\tilde{\boldsymbol{P}}(-\infty),t)\;N_b(\boldsymbol{r},\tilde{\boldsymbol{P}}'(-\infty),t)\right.\right.$$

$$\left. - N_a(\boldsymbol{r},\boldsymbol{p},t)\,N_b(\boldsymbol{r},\boldsymbol{p}',t)\right] - \left(\frac{\partial}{\partial t} + \frac{\boldsymbol{v}+\boldsymbol{v}'}{2}\cdot\frac{\partial}{\partial \boldsymbol{r}} + \boldsymbol{F}_{a0}\cdot\frac{\partial}{\partial \boldsymbol{p}}\right)$$

$$\left.\cdot\int_0^\infty d\tau\;\tau\frac{d}{d\tau}\,N_a(\boldsymbol{r},\tilde{\boldsymbol{P}}(-\tau),t)\;N_b(\boldsymbol{r},\tilde{\boldsymbol{P}}'(-\infty),t)\right\}. \tag{61.3}$$

These equations take into account collective effects; they cannot be brought into the Boltzmann form.

For later use we may also introduce an alternative form of (61.2):

$$\left[\frac{\partial}{\partial t} + \boldsymbol{v}\cdot\frac{\partial}{\partial \boldsymbol{r}} + \left(e_a\tilde{E} + \boldsymbol{F}_{a0}\right)\cdot\frac{\partial}{\partial \boldsymbol{p}}\right] N_a(x,t)$$

$$= \sum_b \int dx'\;\frac{\partial \Phi_{ab}}{\partial \boldsymbol{r}}\cdot\frac{\partial}{\partial \boldsymbol{p}}\,N_a(\boldsymbol{r},\tilde{\boldsymbol{P}}(-\infty),t)\,N_b(\boldsymbol{r},\tilde{\boldsymbol{P}}'(-\infty),t)\,. \tag{61.4}$$

The electric field $\tilde{E}(\boldsymbol{r},t)$ introduced here is defined by the following equations, for a Coulomb plasma:

$$\nabla\times\tilde{E} = 0\,,\qquad \nabla\cdot\tilde{E} = \sum_a 4\pi e_a \int d\boldsymbol{p}\;N_a(\boldsymbol{r},\boldsymbol{p},t) \tag{61.5}$$

or else, replacing the Poisson equation by the equation for the current :

$$\nabla \times \tilde{\boldsymbol{E}} = 0 , \quad \frac{\partial \tilde{\boldsymbol{E}}}{\partial t} = - 4\pi \sum_a e_a \int d\boldsymbol{p}\, \boldsymbol{v}\, N_a(\boldsymbol{r},\boldsymbol{p},t). \tag{61.6}$$

The tilde on $\tilde{\boldsymbol{E}}$ indicates that the source in (61.5), (61.6) is the smoothed phase density.

We now compare these equations, (61.4), (61.5) with Eqs (25.1) and (25.2) for the microscopic phase density $N_a(x,t)$ and the microscopic electric field $\boldsymbol{E}^{\mathrm{M}}(\boldsymbol{r},t)$. The latter are exact microscopic equations for a Coulomb plasma. In the former, we included already the contribution of the short-range correlation. On the right-hand side of (61.4): they are less general, as they only describe long-range fluctuations.

The form (61.4), (61.5) of the equations is convenient, particularly for their relativistic generalization. In the latter case, we substitute in (61.4) :

$$e_a \tilde{\boldsymbol{E}} + \boldsymbol{F}_{a0} \longrightarrow e_a \left[\tilde{\boldsymbol{E}} + \frac{1}{c} (\boldsymbol{v} \times \tilde{\boldsymbol{B}}) \right] + \boldsymbol{F}_{ac} \tag{61.7}$$

we replace on the right-hand side the collision integral by the appropriate relativistic one, and use instead of (61.5) the Maxwell equations for the smoothed fields $\tilde{\boldsymbol{E}}$ and $\tilde{\boldsymbol{B}}$.

62. THE METHOD OF MOMENTS. THE POLARIZATION APPROXIMATION

On the basis of (61.3) or (61.4), (61.5), we may construct a hierarchy of equations for the moments of the random functions $\tilde{N}_a$, $\tilde{\boldsymbol{E}}$, as in section 22. We consider the first equation of this hierarchy.

We use the definition of the one-particle distribution function f_a in terms of the averaged smoothed fields (see (26.1)) :

$$n_a f_a = \langle \tilde{N}_a \rangle, \quad \boldsymbol{E} = \langle \tilde{\boldsymbol{E}} \rangle \tag{62.1}$$

and the identity

$$\left\langle \tilde{N}_a(x,t)\, \tilde{\boldsymbol{E}}(\boldsymbol{r},t) \right\rangle = n_a f_a \boldsymbol{E} + \left\langle \delta\tilde{N}_a\, \delta\tilde{\boldsymbol{E}} \right\rangle_{x,\boldsymbol{r},t} \tag{62.2}$$

where $\delta\tilde{N}_a = \tilde{N}_a - n_a f_a$, $\delta\tilde{\boldsymbol{E}} = \tilde{\boldsymbol{E}} - \boldsymbol{E}$. Thus, by averaging (61.4), (61.5), we find the equations for f_a, $\boldsymbol{E}$:

$$\left[\frac{\partial}{\partial t} + \boldsymbol{v} \cdot \frac{\partial}{\partial \boldsymbol{r}} + \left(e_a \boldsymbol{E} + \boldsymbol{F}_{a0} \right) \cdot \frac{\partial}{\partial \boldsymbol{p}} \right] f_a = I_a^{\mathbf{B}} + \tilde{I}_a$$

$$\nabla \times \boldsymbol{E} = 0 , \quad \nabla \cdot \boldsymbol{E} = \sum_a 4\pi e_a n_a \int d\boldsymbol{p}\, f_a \tag{62.3}$$

where

$$I_a^{\mathbf{B}} = \sum_b n_b \int dx' \,\frac{\partial \Phi_{ab}}{\partial \boldsymbol{r}} \cdot \frac{\partial}{\partial \boldsymbol{p}} f_a(\boldsymbol{r}, \tilde{\boldsymbol{P}}(-\infty), t)\; f_b(\boldsymbol{r}, \tilde{\boldsymbol{P}}'(-\infty), t) \tag{62.4}$$

is the part of the collision integral involving short-range fluctuations, and

$$\tilde{I}_a = -\frac{e_a}{n_a}\frac{\partial}{\partial \boldsymbol{p}} \cdot \left\langle \delta\tilde{N}_a \, \delta\tilde{\boldsymbol{E}} \right\rangle_{x,\boldsymbol{r},t} \tag{62.5}$$

is the one involving long-range fluctuations. In (62.4) the term involving $\tilde{g}_2(\boldsymbol{r}, \tilde{\boldsymbol{P}}(-\infty), \tilde{\boldsymbol{P}}'(-\infty), t)$ disappeared, as the contribution of the long-range fluctuations at short distance $(r < r_{\mathrm{D}})$ is small.

In order to stress the analogy with the case of gases, we shall call the integral $I_a^{\mathbf{B}}$ the Boltzmann collision integral. This is not quite accurate: $I_a^{\mathbf{B}}$ involves not only binary, but also collective collisions (through the effective potential).

The set (62.3) is not closed, as it involves, besides f_a and $\boldsymbol{E}$, the second moment $\langle \delta\tilde{N}_a \, \delta\tilde{E} \rangle$. The equation for the second moment involves the third one, and so on. We have a hierarchy analogous to the one for the smoothed functions f_1, f_2, f_3 (see sects 21, 22).

We now consider, as in section 22, the equations for the moments in the polarization approximation. The equation for the one-time moment $\langle \delta N_a \, \delta N_b \rangle_{x,x',t}$ is obtained as for (22.21) (we now delete again the tilde on $\delta\tilde{N}_a \, \delta\tilde{N}_b$) :

$$\left(\hat{L}_{ab} + \delta\hat{I}_{ax} + \delta\hat{I}_{bx'}\right) \left\langle \delta N_a \, \delta N_b \right\rangle_{x,x',t} + e_a \left\langle \delta \boldsymbol{E} \, \delta N_b \right\rangle_{\boldsymbol{r},x',t} \cdot \frac{\partial n_a f_a}{\partial \boldsymbol{p}}$$

$$+ e_b \left\langle \delta N_a \, \delta \boldsymbol{E} \right\rangle_{x,\boldsymbol{r}',t} \cdot \frac{\partial n_b f_b}{\partial \boldsymbol{p}'} = A_{ab}(x,x',t) \tag{62.6}$$

Here the function :

$$A_{ab}(x,x',t) = \left(\hat{L}_{ab} + \delta\hat{I}_{ax} + \delta\hat{I}_{bx'}\right) n_a \delta_{ab} \left[\delta(x-x') f_a - \frac{1}{V} f_a f_b\right]$$

$$+ \left\{ n_a n_b \overbrace{\hat{\theta}_{ab} f_a(\tilde{\boldsymbol{P}}, t) f_b(\tilde{\boldsymbol{P}}', t)} \right.$$

$$\left. - V^{-1} \delta_{ab} n_a \left[I_a^{\mathbf{B}}(x,t) f_b(x',t) + I_b^{\mathbf{B}}(x',t) f_a(x,t) \right] \right\} \tag{62.7}$$

is defined in terms of the one-particle functions f_a; it thus plays the role of a source term. The operators $\hat{L}_{ab}$, $\hat{\theta}_{ab}$ are defined by (26.4) and (27.5) : $\boldsymbol{P} = \boldsymbol{P}(-\infty)$, $\boldsymbol{P}' = \boldsymbol{P}'(-\infty)$; $I_a^{\mathbf{B}}$ is defined by (62.4), and the action of δI_{ax} on an arbitrary function $F_a(x,t)$ is defined as (see (22.37)) :

$$\delta\hat{I}_{ax} F_a(x,t) = -\sum_c n_c \int dx'' \, \hat{\theta}_{ac} \left[f_a(\boldsymbol{r}, \tilde{\boldsymbol{P}}, t) \, F_c(\boldsymbol{r}, \tilde{\boldsymbol{P}}'', t) \right.$$

$$\left. + F_a(\boldsymbol{r}, \tilde{\boldsymbol{P}}, t) \, f_c(\boldsymbol{r}, \boldsymbol{P}'', t) \right] . \tag{62.8}$$

The equation for the two-time moment $\langle \delta N_a \, \delta N_b \rangle_{x,t,x',t'}$ in the

polarization approximation is analogous to (22.40) :

$$\left(\hat{L}_a + \delta\hat{I}_{ax}\right)\left\langle\delta N_a\ \delta N_b\right\rangle_{x,t,x',t'} + e_a \left\langle\delta \boldsymbol{E}\ \delta N_b\right\rangle_{\boldsymbol{r},t,x',t'} \cdot \frac{\partial\, n_a f_a(x,t)}{\partial \boldsymbol{p}} = 0 . \tag{62.9}$$

The solution of this equation is expressed in terms of the one-time moment, i.e., the initial condition. Thus, for the determination of the spectral density of the long-range fluctuations we need the solution of (62.6). We saw in section 22 that this step can be avoided if we use directly the equations for δN_a, which are equivalent to the set (62.6), (62.9). By analogy to (22.41) we may then write

$$\left(\hat{L}_a + \delta\hat{I}_{ax}\right)\left\{\delta N_a(x,t) - \delta N_a^{\mathbf{source}}(x,t)\right\} + e_a\ \delta\boldsymbol{E}(\boldsymbol{r},t)\cdot\frac{\partial n_a f_a}{\partial \boldsymbol{p}} = 0$$

$$\nabla\times\delta\boldsymbol{E} = 0 , \quad \nabla\cdot\delta\boldsymbol{E} = 4\pi\sum_a e_a\int d\boldsymbol{p}\ \delta N_a \tag{62.10}$$

where the tilde on δN_a, $\delta\boldsymbol{E}$ is omitted. The correlation of the source fluctuations $\delta N_a^{\mathbf{source}}$ is defined by an equation analogous to (22.42) :

$$\left(\hat{L}_a + \delta\hat{I}_{ax}\right)\left\langle\delta N_a\ \delta N_b\right\rangle_{x,t,x',t'} = 0 . \tag{62.11}$$

This equation must be solved with the initial condition

$$\left\langle\delta N_a\ \delta N_b\right\rangle^{\mathbf{source}}_{x,t,x',t'}\Big|_{t=t'} = \left\langle\delta N_a\ \delta N_b\right\rangle^{\mathbf{source}}_{x,x',t} .$$

The function $\left\langle\delta N_a\ \delta N_b\right\rangle^{\mathbf{source}}_{x,x',t}$ is in turn defined by an equation analogous to (22.43) :

$$\left(\hat{L}_{ab} + \delta\hat{I}_{ax} + \delta\hat{I}_{bx'}\right)\left\langle\delta N_a\ \delta N_b\right\rangle^{\mathbf{source}}_{x,x',t} = A_{ab}(x,x',t) \tag{62.12}$$

where $A_{ab}(x,x',t)$ is given by (62.7). This equation is much simpler than (62.6), as it does not involve polarization terms, proportional to $\left\langle\delta\boldsymbol{E}\ \delta N_b\right\rangle$, $\left\langle\delta N_a\ \delta\boldsymbol{E}\right\rangle$.

In the local equilibrium approximation, the terms enclosed in square brackets in (62.7) vanish. Thus, from (62.12), (62.7) follows that the initial condition for (62.11) can be written in the form

$$\left\langle\delta N_a\ \delta N_b\right\rangle^{\mathbf{source}}_{x,x',t} = n_a\ \delta_{ab}\left[\delta(x-x')\ f_a - V^{-1}\ f_a(x,t)\ f_b(x',t)\right] . \tag{62.13}$$

We now consider the approximation in which we neglect the polarization due to the long-range fluctuations of $\delta\boldsymbol{E}$. In this approximation, we omit in (62.6) the terms in $\left\langle\delta\boldsymbol{E}\ \delta N_b\right\rangle$, $\left\langle\delta N_a\ \delta\boldsymbol{E}\right\rangle$ and thus Eqs (62.6) and (62.12) coincide so that

$$\left\langle\delta N_a\ \delta N_b\right\rangle_{x,x',t} = \left\langle\delta N_a\ \delta N_b\right\rangle^{\mathbf{source}}_{x,x',t} \tag{62.14}$$

In the same approximation, we may neglect in (62.10) the term in $\delta \boldsymbol{E}$. Thus $\delta N_a = \delta N_a^{\text{source}}$ and we have

$$\left\langle \delta N_a \ \delta N_b \right\rangle_{x,t,x',t'} = \left\langle \delta N_a \ \delta N_b \right\rangle^{\text{source}}_{x,t,x',t'} . \tag{62.15}$$

Instead of Eqs (62.11), (62.12) we may use the equivalent Langevin equation

$$\left(\hat{L}_a + \delta \hat{I}_{ax}\right) \delta N_a^{\text{source}} = y_a(x,t) \tag{62.16}$$

which represent the linearized Boltzmann equation of the plasma. In the next section we establish the relation between the correlations of the random sources y_a of the Langevin equation and the function $A_{ab}(x,x',t)$, i.e. the source in the equation for the moment $\langle \delta N_a \ \delta N_b \rangle_{x,x',t}$.

As in section 22, the function A_{ab} can be represented as the sum of two terms

$$A_{ab} = \tilde{A}_{ab} + A^{\mathbf{B}}_{ab} . \tag{62.17}$$

The first term represents the long-range fluctuations, and tends to zero in equilibrium. It is defined by an equation analogous to (22.23) :

$$\tilde{A}_{ab}(x,x',t) = n_a \ \delta_{ab} \Big\{ \delta(x-x') \ \tilde{I}_a(x,t) - V^{-1} \Big[\tilde{I}_a(x,t) \ f_b(x',t) + \tilde{I}_b(x',t) \ f_a(x,t) \Big] \Big\} . \tag{62.18}$$

It is expressed in terms of the collision integral $\tilde{I}_a$. An explicit expression will be found in section 64.

The second term in (62.17) is defined by a formula analogous to $(22.27)_3$:

$$A^{\mathbf{B}}_{ab}(x,\ x',\ t) = \left[\left(\delta\hat{I}_{ax} + \delta\hat{I}_{bx'} \right) - \left(\delta\hat{I}_{ax} + \delta\hat{I}_{bx'} \right)_0 \right] n_a \, \delta_{ab} \, \delta(x-x') \, f_a(x,t) . \tag{62.19}$$

This part of A_{ab} is due to the discreteness of the collision events and remains non-zero even in equilibrium.

63. THE SPECTRAL DENSITIES OF THE LONG-RANGE FLUCTUATIONS

We consider the spectral densities of the long-range fluctuations of an ideal plasma, i.e., we neglect retardation and inhomogeneity effects. We first look for the spectral function of the source fluctuations, and from these, through Eqs (62.10), the spectral densities of the long-range fluctuations δN_a, $\delta \boldsymbol{E}$. We note that (62.11), (62.12) are analogous to (22.44), (22.45). Because of the analogy, we may write directly the solution of the former, corresponding to (23.6) :

$$\left(\delta N_a \ \delta N_b\right)^{\text{source}}_{\omega,\boldsymbol{k},\boldsymbol{p},\boldsymbol{p}'} = \frac{A_{ab}(\boldsymbol{k},\boldsymbol{p},\boldsymbol{p}')}{[-i(\omega - \boldsymbol{k}\cdot\boldsymbol{v}) + \not{I}_{a\boldsymbol{p}}]\,[i(\omega - \boldsymbol{k}\cdot\boldsymbol{v}') + \not{I}_{b\boldsymbol{p}'}]} \tag{63.1}$$

where we introduced the notation

$$\mathcal{I}_{ap} = F_{a0} \cdot \frac{\partial}{\partial p} + \delta \hat{I}_{ap} \tag{63.2}$$

corresponding to (23.3). The operator $\delta \hat{I}_{ap}$ coincides with (62.8). We do not consider the dependence on r and t as these variables enter the collision integral only as parameters in the ideal case.

The expression of the spatial Fourier transform of A_{ab} is obtained from (62.7) :

$$\begin{aligned} A_{ab}(k, p, p') = \left(\hat{\mathcal{I}}_{ap} + \hat{\mathcal{I}}_{bp'}\right) n \delta_{ab}\left[\delta(p-p')\, f_a - f_a(p)\, f_b(p')\right] \\ + \left\{n_a n_b \int dr\, \hat{\theta}_{ab}\, f_a(\tilde{P}) f_b(\tilde{P}') \right. \\ \left. - n_a \delta_{ab}\left[I_a^{\mathrm{B}}(p)\, f_b(p') + I_b^{\mathrm{B}}(p')\, f_a(p)\right]\right\} . \end{aligned} \tag{63.3$_1$}$$

In order to obtain this result we note that, to zeroth order in $kl_{ph} \sim kr_{\mathrm{D}}$, the Fourier transform of the expression

$$\hat{\theta}_{ab}\, f_a(\tilde{P})\, f_b(\tilde{P}')$$

is independent of k (see (23.4)). Furthermore, we used the relation

$$(2\pi)^3\, \delta(k=0)/V = 1$$

which is used in the transformation of the second and the fourth term of (62.7). (63.3) has the properties :

$$\int dp\, A_{ab}(k, p, p') = \int dp'\, A_{ab}(k, p, p') = 0 . \tag{63.4}$$

Substituting (63.1) into the corresponding solution of the Langevin equation (62.16) we find the relation

$$(y_a y_b)_{\omega, k, p, p'} = A_{ab}(k, p, p') . \tag{63.5}$$

This equation corresponds to (23.8). As the right-hand side is independent of ω and, for $kr_{\mathrm{D}} \ll 1$, also of k, the random source in the Langevin equation is δ-correlated in both space and time.

In the local equilibrium approximation, the term in braces in (63.3)$_1$ vanishes, and

$$A_{ab}(k, p, p') = (\mathcal{I}_{ap} + \mathcal{I}_{bp'})\, n_a\, \delta_{ab}\left[\delta(p-p')\, f_a(p) - f_a f_b\right] \tag{63.3$_2$}$$

Thus, from (63.1) :

$$\left(\delta N_a\, \delta N_b\right)^{\text{source}}_{\omega, k, p, p'} = \frac{(\hat{\mathcal{I}}_{ap} + \hat{I}_{bp'})\, n_a \delta_{ab}[\delta(p-p') f_a(p) - f_a f_b]}{[\omega - k\cdot v + i\, \hat{\mathcal{I}}_{ap}]\, [(\omega - k\cdot v') - i\, \mathcal{I}_{bp'}]} . \tag{63.6}$$

We now consider Eqs (62.10) in order to determine the spectral density of $\delta\tilde{N}_a, \delta\tilde{E}$. From the first of these equations we find

$$\delta N_a(\omega, k, p, t) = \delta N_a^{\text{source}}(\omega, k, p, t) - \frac{i e_a n_a}{\omega - k\cdot v + i\,\mathcal{I}_{ap}}\, \delta E(\omega, k) \cdot \frac{\partial f_a(p)}{\partial p} . \tag{63.7}$$

From the field equations we find

$$\delta \boldsymbol{E}(\omega, \boldsymbol{k}) + \frac{i\boldsymbol{k}}{k^2} \sum_a 4\pi e_a \int d\boldsymbol{p}\, \delta N_a(\omega, \boldsymbol{k}, \boldsymbol{p})$$
$$= -\frac{i\boldsymbol{k}}{k^2} \sum_a 4\pi e_a \int d\boldsymbol{p}\, \delta N_a^{\text{source}}(\omega, \boldsymbol{k}, \boldsymbol{p}) \equiv \delta \boldsymbol{E}^{\text{source}}(\omega, \boldsymbol{k}) . \qquad (63.8)$$

The set of equations (63.7), (63.8) differs from (35.2), (34.12) by the substitution

$$\frac{1}{\omega - \boldsymbol{k}\cdot\boldsymbol{v} + i\Delta} \longrightarrow \frac{1}{\omega - \boldsymbol{k}\cdot\boldsymbol{v} + i\hat{I}_{a\boldsymbol{p}}} \equiv \hat{L}^{-1}_{a,\omega,\boldsymbol{k},\boldsymbol{p}} . \qquad (63.9)$$

Thus, we find again an expression of the form

$$\left(\delta\tilde{\boldsymbol{E}}\cdot\delta\tilde{\boldsymbol{E}}\right)_{\omega,\boldsymbol{k}} = \frac{(\delta\tilde{\boldsymbol{E}}\cdot\delta\tilde{\boldsymbol{E}})^{\text{source}}_{\omega,\boldsymbol{k}}}{|\tilde{\varepsilon}(\omega,\boldsymbol{k})|^2} \qquad (63.10)$$

similar to (35.6), but the dielectric constant and the spectral density of the source functuations are defined differently :

$$\tilde{\varepsilon}(\omega,\boldsymbol{k}) = 1 + \sum_a \frac{4\pi e_a^2 n_a}{k^2} \int d\boldsymbol{p}\, \hat{L}^{-1}_{a,\omega,\boldsymbol{k},\boldsymbol{p}}\, \boldsymbol{k}\cdot\frac{\partial f_a}{\partial \boldsymbol{p}} \qquad (63.11)$$

$$\left(\delta\tilde{\boldsymbol{E}}\cdot\delta\tilde{\boldsymbol{E}}\right)^{\text{source}}_{\omega,\boldsymbol{k}} = \sum_a\sum_b \frac{(4\pi)^2 e_a e_b}{k^2} \int d\boldsymbol{p}\, d\boldsymbol{p}'\, \hat{L}^{-1}_{a,\omega,\boldsymbol{k},\boldsymbol{p}}\, \hat{L}^{-1*}_{b,\omega,\boldsymbol{k},\boldsymbol{p}'}\, A_{ab}(\boldsymbol{p},\boldsymbol{p}') . \qquad (63.12)_1$$

These equations follow from (63.8) and (63.1). In local equilibrium $(63.12)_1$ reduces to the much simpler form

$$\left(\delta\tilde{\boldsymbol{E}}\cdot\delta\tilde{\boldsymbol{E}}\right)^{\text{source}}_{\omega,\boldsymbol{k}} = \sum_a \frac{(4\pi)^2 e_a^2 n_a}{k^2}\, 2\,\mathrm{Re} \int d\boldsymbol{p}\; i\, \hat{L}^{-1}_{a,\omega,\boldsymbol{k},\boldsymbol{p}}\, f_a(\boldsymbol{p}) . \qquad (63.12)_2$$

We transform this expression, noting that in equilibrium

$$\hat{I}_{a\boldsymbol{p}} f_a = \delta\hat{I}_{a\boldsymbol{p}} f_a = 0. \qquad (63.13)$$

We therefore obtain

$$2\,\mathrm{Re} \int d\boldsymbol{p}\; i\hat{L}^{-1}_{a,\omega,\boldsymbol{k},\boldsymbol{p}}\, f_a(\boldsymbol{p})$$
$$= 2\,\mathrm{Re} \int d\boldsymbol{p}\, \frac{i}{\omega - \boldsymbol{k}\cdot\boldsymbol{v} + i\delta\hat{I}_{a\boldsymbol{p}}}\, \frac{\omega + i\delta\hat{I}_{a\boldsymbol{p}}}{\omega}\, f_a$$
$$= 2\,\mathrm{Re} \int d\boldsymbol{p}\, \frac{i\boldsymbol{k}\cdot\boldsymbol{v}}{\omega - \boldsymbol{k}\cdot\boldsymbol{v} + i\delta\hat{I}_{a\boldsymbol{p}}}\, f_a = -2\,\mathrm{Re} \int d\boldsymbol{p}\, \frac{i\boldsymbol{k}\cdot(\partial f_a/\partial\boldsymbol{p})}{\omega - \boldsymbol{k}\cdot\boldsymbol{v} + i\delta\hat{I}_{a\boldsymbol{p}}}\, k_{\mathrm{B}}T \qquad (63.14)$$

Using also the expression of the imaginary part of the dielectric constant derived from (63.11) we finally get

$$\left(\delta\tilde{\boldsymbol{E}}\cdot\delta\tilde{\boldsymbol{E}}\right)_{\omega,\boldsymbol{k}} = \frac{8\pi}{\omega}\, \frac{\mathrm{Im}\,\tilde{\varepsilon}(\omega,\boldsymbol{k})}{|\tilde{\varepsilon}(\omega,\boldsymbol{k})|^2}\, k_{\mathrm{B}}T . \qquad (63.15)$$

This formula, valid in equilibrium, is universal.

We now calculate the spectral density $(\delta\tilde{N}_a\, \delta\tilde{\boldsymbol{E}})_{\omega,\boldsymbol{k},\boldsymbol{p}}$ which defines the

collision integral $\tilde{I}_a$. From (63.7) follows that it can be written as a sum of two terms (we omit the tilde) :

$$\left(\delta N_a\,\delta \boldsymbol{E}\right)_{\omega,\boldsymbol{k},\boldsymbol{p}} = \left(\delta N_a\,\delta \boldsymbol{E}\right)^{\mathrm{ind}}_{\omega,\boldsymbol{k},\boldsymbol{p}} + \left(\delta N_a^{\mathrm{source}}\,\delta \boldsymbol{E}\right)_{\omega,\boldsymbol{k},\boldsymbol{p}} . \tag{63.16}$$

The induced part of the spectral density is defined as

$$\left(\delta N_a\,\delta \boldsymbol{E}\right)^{\mathrm{ind}}_{\omega,\boldsymbol{k},\boldsymbol{p}} = -\frac{i\,e_a n_a}{\omega-\boldsymbol{k}\cdot\boldsymbol{v}+i\hat{I}_{a\boldsymbol{p}}}\left(\delta \boldsymbol{E}\cdot\delta \boldsymbol{E}\right)_{\omega,\boldsymbol{k}}\,\frac{\boldsymbol{k}}{k^2}\left(\boldsymbol{k}\cdot\frac{\partial f_a}{\partial \boldsymbol{p}}\right). \tag{63.17}$$

It involves the spectral density of the field fluctuations. The second term is given by

$$\left(\delta N^{\mathrm{source}}\,\delta \boldsymbol{E}\right)_{\omega,\boldsymbol{k},\boldsymbol{p}} = \frac{1}{\tilde{\varepsilon}^*(\omega,\boldsymbol{k})}\left(\delta N_a\,\delta \boldsymbol{E}\right)^{\mathrm{source}}_{\omega,\boldsymbol{k},\boldsymbol{p}} \tag{63.18}$$

where the dielectric constant is defined by (63.11) and the spectral density by :

$$\left(\delta N_a\,\delta \boldsymbol{E}\right)^{\mathrm{source}}_{\omega,\boldsymbol{k},\boldsymbol{p}} = \frac{i\boldsymbol{k}}{k^2}\sum_b 4\pi e_b\int d\boldsymbol{p}'\,\frac{A_{ab}(\boldsymbol{k},\boldsymbol{p},\boldsymbol{p}')}{[\omega-\boldsymbol{k}\cdot\boldsymbol{v}+i\hat{I}_{a\boldsymbol{p}}]\,[\omega-\boldsymbol{k}\;\;\boldsymbol{v}-i\hat{I}_{b\boldsymbol{p}'}]} . \tag{63.19$_1$}$$

This expression follows from (63.1) and (63.8).

In the local-equilibrium approximation we use instead of (63.1) the simpler formula (63.6). We then find, instead of $(63,19)_1$:

$$\left(\delta N_a\,\delta \boldsymbol{E}\right)^{\mathrm{source}}_{\omega,\boldsymbol{k},\boldsymbol{p}} = \frac{i\boldsymbol{k}}{k^2}\,4\pi e_a n_a\cdot 2\,\mathrm{Re}\,i\hat{L}^{-1}_{\omega,\boldsymbol{k},\boldsymbol{p}}\,f_a(\boldsymbol{p}) . \tag{63.19$_2$}$$

We finally write the expression for the real part of the spectral density (63.16) which defines the collision integral of the ideal plasma. In the local-equilibrium approximation we find from $(63.17)-(63.19)_2$

$$\mathrm{Re}\left(\delta N_a\,\delta \boldsymbol{E}\right)_{\omega,\boldsymbol{k},\boldsymbol{p}} = -e_a n_a\,\mathrm{Re}\,i\,\hat{L}^{-1}_{\omega,\boldsymbol{k},\boldsymbol{p}}\left(\delta E\cdot\delta \boldsymbol{E}\right)_{\omega,\boldsymbol{k}}\,\frac{\boldsymbol{k}}{k^2}\left(\boldsymbol{k}\cdot\frac{\partial f_a}{\partial \boldsymbol{p}}\right)$$
$$-\,4\pi e_a n_a\,\frac{\boldsymbol{k}}{k^2}\,\frac{\mathrm{Im}\,\tilde{\varepsilon}(\omega,\boldsymbol{k})}{|\tilde{\varepsilon}\,(\omega,\boldsymbol{k})|^2}\,2\,\mathrm{Re}\,i\hat{L}^{-1}_{a,\omega,\boldsymbol{k},\boldsymbol{p}}\;f_a(\boldsymbol{p}) . \tag{63.20}$$

If the local-equilibrium approximation is insufficient, we must use $(63.12)_1$ instead of $(63.12)_2$ in the derivation of the spectral density (63.20).

In a similar way we may calculate the more general spectral density $(\delta\tilde{N}_a\,\delta\tilde{N}_b)_{\omega,\boldsymbol{k},\boldsymbol{p},\boldsymbol{p}'}$.

The expressions derived so far for the spectral densities of long-range fluctuations are not explicit, as they involve the operators $\hat{L}^{-1}_{a,\omega,\boldsymbol{k},\boldsymbol{p}}$, which, in turn, involve the operators $\hat{I}_{a\boldsymbol{p}}$, $\delta\hat{I}_{a\boldsymbol{p}}$. The derivation of explicit forms for the spectral densities is thus a difficult problem. In some cases we may use for simplicity model expressions for the collision integrals I^{B}_a. Thus, for instance, in the work of Sitenko and Gurin [54] the Bathnagar-Gross-Krook model was used in the

calculation of the fluctuations in a plasma in equilibrium. The method of calculation used in this work is described in detail in reference [15].

64. KINETIC EQUATIONS FOR PLASMAS WITH ACCOUNT OF THE LONG-RANGE FLUCTUATIONS

We represent the collision integral (62.5) in the form

$$\tilde{I}_a = -\frac{e_a}{n_a}\frac{1}{(2\pi)^4}\frac{\partial}{\partial \boldsymbol{p}}\cdot\int d\omega\, d\boldsymbol{k}\,\mathrm{Re}\left(\delta N_a\,\delta \boldsymbol{E}\right)_{\omega,\boldsymbol{k},\boldsymbol{p}} . \tag{64.1}$$

Using Eq (63.20) we obtain

$$\tilde{I}_a = -\frac{e_a^2}{(2\pi)^4}\frac{\partial}{\partial \boldsymbol{p}}\cdot\int d\omega\, d\boldsymbol{k}\,\frac{\boldsymbol{k}}{k^2}\,\mathrm{Re}\, i\,\hat{L}^{-1}_{a,\omega,\boldsymbol{k},\boldsymbol{p}}\left\{\left(\delta\tilde{E}\cdot\delta\tilde{E}\right)_{\omega,\boldsymbol{k}}\,\boldsymbol{k}\cdot\frac{\partial f_a(\boldsymbol{r},\boldsymbol{p},t)}{\partial \boldsymbol{p}}\right.$$
$$\left.+\frac{8\pi\,\mathrm{Im}\,\tilde{\varepsilon}(\omega,\boldsymbol{k})}{|\tilde{\varepsilon}(\omega,\boldsymbol{k})|^2}\,f_a(\boldsymbol{r},\boldsymbol{p},t)\right\} \tag{64.2}$$

If in the operator $\hat{L}^{-1}$ appearing here we replace $\hat{I}_{a\boldsymbol{p}}$ by Δ and go to the limit $\Delta\to 0$, we recover the Balescu-Lenard collision integral (37.2).

In equilibrium the expression (62.5) vanishes. Indeed, substituting the Maxwell distribution in the integrand of (64.2) and using (63.15) we obtain

$$\left(\delta\tilde{E}\cdot\delta\tilde{E}\right)_{\omega,\boldsymbol{k}}\frac{\boldsymbol{k}}{k^2}\,\mathrm{Re}\, i\,\hat{L}^{-1}_{a,\omega,\boldsymbol{k},\boldsymbol{p}}\left[\boldsymbol{k}\cdot\frac{\partial f_a}{\partial \boldsymbol{p}}+\frac{\omega}{k_B T}f_a\right]$$
$$=\frac{(\delta\tilde{E}\cdot\delta\tilde{E})_{\omega,\boldsymbol{k}}}{k_B T}\frac{\boldsymbol{k}}{k^2}\,\mathrm{Re}\, i\,\hat{L}^{-1}_{a,\omega,\boldsymbol{k},\boldsymbol{p}}(\omega-\boldsymbol{k}\cdot\boldsymbol{v})\,f_a = 0$$

because in equilibrium $\delta\hat{I}_{a\boldsymbol{p}}f_a = 0$.

The collision integral (64.2) has the properties :

$$\sum_a n_a\int d\boldsymbol{p}\,\phi_a(\boldsymbol{p})\,I_a = 0 \quad \text{for} \quad \phi_a = 1,\boldsymbol{p},\frac{p^2}{2m_a} . \tag{64.3}$$

We recall that Eq. (64.2) is valid whenever in Eq. (63.3) the terms enclosed in braces can be neglected and the simpler expression $(63.3)_2$ can be used. This is valid, in particular, for the local equilibrium state. For concrete calculations the complicated operator $\delta\hat{I}_{a\boldsymbol{p}}$ is replaced by some model expression for the Boltzmann collision integral. It is therefore convenient to use (64.2) even for non local-equilibrium states.

Equation (62.4), (64.2) is valid only for the ideal plasma. We now consider its extension to the non-ideal case. In this problem the kinetic equation can also be written in the form :

$$\left[\frac{\partial}{\partial t}+\boldsymbol{v}\cdot\frac{\partial}{\partial \boldsymbol{r}}+\left(e_a\boldsymbol{E}+\boldsymbol{F}_{a0}\right)\cdot\frac{\partial}{\partial \boldsymbol{p}}\right]f_a = I_a^{\mathrm{B}}+\tilde{I}_a . \tag{64.4}$$

However, now the collision integral I_a^{B} involves three terms (58.8), defined by Eqs. (59.7)–(59.9). For the integral $\hat{I}_a$ we must use instead of (62.5), (64.1)

expressions analogous to (58.4), (58.7). Thus,

$$\tilde{I}_a(x,t) = -\frac{e_a}{n_a}\frac{1}{(2\pi)^4}\frac{\partial}{\partial p_i}\int d\omega\, d\boldsymbol{k}\left\{\mathrm{Re}\left(\delta\tilde{N}_a\,\delta\tilde{E}_i\right)_{\omega,\boldsymbol{k},\boldsymbol{p},\boldsymbol{r},t} + \frac{1}{k^2}\left(\frac{1}{2}\delta_{ij} - \frac{k_i k_j}{k^2}\right)\frac{\partial}{\partial r_j}\,\mathrm{Im}\left(\delta\tilde{N}_a\,\boldsymbol{k}\cdot\delta\tilde{\boldsymbol{E}}\right)_{\omega,\boldsymbol{k},\boldsymbol{p},\boldsymbol{r},t}\right\}. \tag{64.5}$$

To obtain an explicit expression for this integral the spectral density

$$\left(\delta N_a\;\delta \boldsymbol{E}\right)_{\omega,\boldsymbol{k},\boldsymbol{p},\boldsymbol{r},t}$$

must be calculated, taking into account (to first order) the retardation and the spatial inhomogeneity.

As Eq. (64.5) involves the spectral density of fluctuations with correlation times $\tau_{\mathbf{corr}} > 1/\omega_{\mathbf{L}}$ these fluctuations can be considered to be plasmons (see sect. 57). The spatially homogeneous plasma can therefore be treated as in section 57; the corresponding collision integral is analogous to (57.8):

$$\tilde{I}_a(\boldsymbol{p},t) = \frac{e_a^2}{(2\pi)^4}\frac{\partial}{\partial \boldsymbol{p}}\cdot\int d\boldsymbol{k}\, d\omega\,\frac{\boldsymbol{k}}{k^2}\,\mathrm{Re}\; i\,\hat{L}^{-1}_{a,\omega,\boldsymbol{k},\boldsymbol{p}}\left\{\left(\delta\tilde{\boldsymbol{E}}\cdot\delta\tilde{\boldsymbol{E}}\right)_{\omega,\boldsymbol{k},t-\frac{\tau}{2}}\cdot\boldsymbol{k}\cdot\frac{\partial f_a(\boldsymbol{p},t-\tau)}{\partial\boldsymbol{p}} + \frac{8\pi\;\mathrm{Im}\,\tilde{\varepsilon}(\omega,\boldsymbol{k})}{|\tilde{\varepsilon}(\omega,\boldsymbol{k})|^2}\,f_a(\boldsymbol{p},t-\tau)\right\}. \tag{64.6}$$

As in (57.8), only zeroth and first order corrections in the retardation are meaningful.

The spectral density of fluctuations appearing in (64.6) obeys an equation analogous to (57.15) :

$$\frac{\partial}{\partial t}\left(\delta\tilde{\boldsymbol{E}}\cdot\delta\tilde{\boldsymbol{E}}\right)_{\omega,\boldsymbol{k},t} = -2\tilde{\gamma}\left[\left(\delta\tilde{\boldsymbol{E}}\cdot\delta\tilde{\boldsymbol{E}}\right)_{\omega,\boldsymbol{k},t} - \frac{(\delta\tilde{\boldsymbol{E}}\cdot\delta\tilde{\boldsymbol{E}})^{\mathbf{source}}_{\omega,\boldsymbol{k},t}}{|\tilde{\varepsilon}(\omega,\boldsymbol{k})|^2}\right] - 2\,\frac{\partial^2\,\mathrm{Re}\,\tilde{\varepsilon}}{\partial t\,\partial\omega}\left[\frac{\partial\,\mathrm{Re}\,\tilde{\varepsilon}}{\partial\omega}\right]^{-1}\left(\delta\tilde{\boldsymbol{E}}\cdot\delta\tilde{\boldsymbol{E}}\right)_{\omega,\boldsymbol{k},t} \tag{64.7}$$

where

$$\tilde{\gamma} = \mathrm{Im}\,\tilde{\varepsilon}\left(\frac{\partial\,\mathrm{Re}\,\tilde{\varepsilon}}{\partial\omega}\right)^{-1}, \qquad \mathrm{Re}\,\tilde{\varepsilon}\,(\omega,\boldsymbol{k}) = 0\;. \tag{64.8}$$

are respectively the damping rate and the dispersion relation for the long-range fluctuations.

If the time-dependence of the frequency is taken into account in (64.7), there appears an extra term analogous to (57.16). Such a term appears also in the kinetic equation.

Taking into account the spatial inhomogeneity, we get two additional contributions to $\tilde{I}_a$. One of them is defined by the second term of (64.5). The

spectral density appearing in it must be evaluated only to zeroth order in the retardation and in the inhomogeneity; see Eqs (63.16) – $(63.19)_2$. The second additional term is related to the contribution of the inhomogeneity of the spectral density $(\delta N_a \delta \boldsymbol{E})_{\omega,\boldsymbol{k},\boldsymbol{p},\boldsymbol{r},t}$ in the first term of (64.5).

65. HYDRODYNAMICAL EQUATIONS WITH ACCOUNT OF THE LONG-RANGE FLUCTUATIONS

By taking into account the long-range fluctuations, new terms appear in the hydrodynamical equations of the plasma, both in the thermodynamic functions as well as in the transport coefficients.

Thus, for instance, instead of (60.20) for the contribution of the interactions to the internal energy density we obtain

$$\Delta U^a = \sum_a \sum_b \frac{1}{2} \frac{1}{(2\pi)^3} \left\{ \int_{k > r_{\mathrm{D}}^{-1}} d\boldsymbol{k}\, \nu_{ab}(\boldsymbol{k}) \left(\delta n_a\, \delta n_b\right)_{\boldsymbol{k},\boldsymbol{r},t} + \int_{k < r_{\mathrm{D}}^{-1}} d\boldsymbol{k}\, \nu_{ab}(\boldsymbol{k}) \left(\delta \tilde{n}_a\, \delta \tilde{n}_b\right)_{\boldsymbol{k},\boldsymbol{r},t} \right\} . \tag{65.1}$$

The second term takes into account the long-range fluctuations.

In equilibrium the two integrals are combined, and we get the known result. The contribution of the second term is important whenever the state differs significantly from equilibrium in the spectral region $k < 1/r_{\mathrm{D}}$.

Expressions analogous to (65.1) can be written also for the other thermodynamic functions.

As a result of the structure of the collision integral

$$I_a = I_a^{\mathrm{B}} + \tilde{I}_a \tag{65.2}$$

the viscosity coefficient can be written in the form

$$\eta_a \sim \frac{n_a k_{\mathrm{B}} T_a}{\nu_a^{\mathrm{B}} + \tilde{\nu}_a} \tag{65.3}$$

where ν_a^{B} is the collision frequency for the itegral I_a^{B} and $\tilde{\nu}_a$ the corresponding quantity for $\tilde{I}_a$.

For states close to equilibrium (65.3) corresponds to the ordinary expression for the viscosity. If in the region of long scales $(k < 1/r_{\mathrm{D}})$ there are strong deviations from equilibrium due for instance to turbulence, we may have a change in sign of $\tilde{\nu}_a$. Thus the viscosity can be larger.

It is impossible to give here concrete examples: we refer the reader to the literature [19-24, 26-30].

Note that the problem of the division of the fluctuations into short- and long-range ones appears also in the theory of collisionless plasmas. The correlation time of the short-range fluctuations is defined, for instance, by the Landau

damping. The correlation time of the long-range fluctuations can be defined by the time of flight, i.e. the life-time of a particle in the volume under consideration, by the dissipative processes at the boundary, etc.

The kinetic fluctuations play an important rôle in various systems. They define, for instance, the essential fluctuations of laser radiation and of molecular generators [56-58]. The laser fluctuations are similar to those of a collisional plasma, and those of a molecular generator to those of a collisionless plasma. We further note that the theory of long-range fluctuations is presently intensively developed for the study of electron-phonon systems [59-62].

PART III

Quantum Kinetic Equations for Nonideal Gases and Nonideal Plasmas

INTRODUCTION

The last part of this book is devoted to the kinetic theory of the nonideal quantum gases and quantum plasmas. The name 'nonideal' means, as before, that within a given model (pair-collision approximation, polarization approximation, etc.) we consider the contribution of the interactions to both the dissipative and the non-dissipative quantities.

The first attempt at a construction of a quantum Boltzmann equation for a nonideal gas is found in the work of Green [1] but it was not very successful. More elaborate kinetic theories of this type are found in the work of Kadanoff and Baym [2], Baerwinkel and Grossmann [3,4], and Klimontovich and Ebeling [5].

Quantum kinetic equations for a nonideal, fully ionized, non-degenerate plasma were considered in the work of Klimontovich and Ebeling [5]. In the paper of Klimontovich and Kraeft [6] the exchange effects in these systems were considered.

The kinetic processes in an ideal quantum plasma are usually described by the quantum analogue of the Balescu-Lenard equation [7,8]. As in the classical case this equation takes account of the dynamical polarization which screens the interactions at large distances. The contribution of the short-range interactions is considered only in the weak coupling approximation (Born approximation).

For nonideal plasmas the weak coupling approximation is insufficient, as the short-range interactions are quite important in the thermodynamic quantities. Thus, for short distances we must use the binary collision approximation.

We already know that the simultaneous treatment of the long-range dynamical polarization effects and of the strong short-range interactions in a nonideal plasma leads to quite complicated equations. Therefore, as in sections 55, 56 we will only use the average dynamical polarization. However, in the treatment of the short-range

effects we retain all terms of the perturbation series in the interactions. We thus account for the strong interactions of the charged particles.

A much more difficult problem is the derivation of kinetic equations for chemically reacting systems and for partially ionized plasmas. This is due to the rôle played by the internal degrees of freedom of the free atoms and of the molecules. The kinetic theory of these systems started its development not long ago: it is in a much less advanced state than the kinetic theory of gases and of fully ionized plasmas. A short account of these problems is given in the last chapter of this book. It concludes the book by showing possible ways of generalizing the theory developed here and by calling attention to these very important and interesting problems. A fuller treatment of these problems could only be given in another book.

CHAPTER 12

Quantum Kinetic Equations for Nonideal Gases

66. HIERARCHY OF EQUATIONS FOR THE QUANTUM DISTRIBUTION FUNCTIONS. THE BINARY COLLISION APPROXIMATION

The one-particle, two-particle , ... density matrices of a many-component gas will be denoted by $\rho_a(\boldsymbol{r}',\boldsymbol{r}'',t)$, $\rho_{ab}(\boldsymbol{r}_1',\boldsymbol{r}_2',\boldsymbol{r}_1'',\boldsymbol{r}_2'',t)$, ... We denote by f_a, f_b, ... the corresponding quantum distribution functions, i.e. the density matrices in the Wigner representation [9 – 12]†. The functions ρ_a and f_a are interrelated in the following way

$$f_a(\boldsymbol{r},\boldsymbol{p},t) = \frac{1}{(2\pi)^3}\int d\boldsymbol{\Upsilon}\ \rho_a(\boldsymbol{r}+\tfrac{1}{2}\hbar\boldsymbol{\Upsilon},\ \boldsymbol{r}-\tfrac{1}{2}\hbar\boldsymbol{\Upsilon})\ e^{-i\boldsymbol{\Upsilon}\cdot\boldsymbol{p}}\ \frac{(2\pi\hbar)^3}{V}\,. \tag{66.1}$$

The normalization conditions are:

$$\frac{1}{(2\pi\hbar)^3}\int d\boldsymbol{r}\ d\boldsymbol{p}\ f_a(\boldsymbol{r},\boldsymbol{p},t) = 1\,,\quad \frac{1}{V}\int d\boldsymbol{r}\ \rho_a(\boldsymbol{r},\boldsymbol{r},t) = 1\,. \tag{66.2}$$

Similar relations hold for the functions ρ_{ab}, f_{ab}; $\rho_{abc}, f_{abc}, \ldots$ For the quantum distribution functions, or for the density matrices, we may derive a hierarchy of equations similar to the BBGKY hierarchy. The first equation in this hierarchy is [10 – 12] :

$$\begin{aligned}\frac{\partial f_a}{\partial t} + \boldsymbol{v}_1\cdot\frac{\partial f_a}{\partial \boldsymbol{r}_1} - \frac{i}{\hbar}\int d\boldsymbol{\Upsilon}\ d\boldsymbol{\eta}\Big[U_{a0}(\boldsymbol{r}_1-\tfrac{1}{2}\hbar\boldsymbol{\Upsilon}) - U_{a0}(\boldsymbol{r}_1+\tfrac{1}{2}\hbar\boldsymbol{\Upsilon})\Big]\\ \cdot e^{i\boldsymbol{\Upsilon}\cdot(\boldsymbol{\eta}-\boldsymbol{p}_1)} f(\boldsymbol{r}_1,\boldsymbol{\eta},t) = \frac{i}{\hbar}(2\pi)^{-3}\sum_b N_b\frac{1}{(2\pi\hbar)^3}\int d\boldsymbol{r}_2\ d\boldsymbol{p}_2\,d\boldsymbol{\Upsilon}\ d\boldsymbol{\eta}\\ \Big[\Phi_{ab}(|\boldsymbol{r}_1-\boldsymbol{r}_2-\tfrac{1}{2}\hbar\boldsymbol{\Upsilon}|)-\Phi_{ab}(|\boldsymbol{r}_1-\boldsymbol{r}_2+\tfrac{1}{2}\hbar\boldsymbol{\Upsilon}|)\Big]\, e^{i\boldsymbol{\Upsilon}\cdot(\boldsymbol{\eta}-\boldsymbol{p}_1)}\ f_{ab}(\boldsymbol{r}_1,\boldsymbol{\eta},\boldsymbol{r}_2,\boldsymbol{p}_2)\end{aligned} \tag{66.3}$$

† A detailed discussion of Wigner functions appears in reference [47]. (Transl.)

Here U_{a0} is the potential of the external field. For $\hbar \to 0$, this equation reduces to

$$\frac{\partial f_a}{\partial t} + \boldsymbol{v}_1 \cdot \frac{\partial f_a}{\partial \boldsymbol{r}_1} + \boldsymbol{F}_{a0} \cdot \frac{\partial f_a}{\partial \boldsymbol{p}_1} = \sum_b N_b \frac{1}{(2\pi\hbar)^3} \int d\boldsymbol{r}_2\, d\boldsymbol{p}_2 \frac{\partial \Phi_{ab}}{\partial \boldsymbol{r}_1} \cdot \frac{\partial f_{ab}}{\partial \boldsymbol{p}_1} . \qquad (66.4)$$

For the one-component gas, this equation is identical to (1.19). For comparison we must note the different normalization of the classical and of the quantum distribution functions. From (1.14) and (66.2) we have

$$\frac{V}{(2\pi\hbar)^3} f_a^{\mathrm{QU}} = f_a^{\mathrm{CL}} , \qquad \frac{V^2}{(2\pi\hbar)^6} f_{ab}^{\mathrm{QU}} = f_{ab}^{\mathrm{CL}} . \qquad (66.5)$$

In the forthcoming developments it is usually more convenient to use the density matrix ρ_{ab} rather than f_{ab}. By analogy with (2.4) we introduce the correlation matrix as follows :

$$\rho_{ab} = \rho_a \rho_b + g_{ab}(\boldsymbol{r}_1', \boldsymbol{r}_2', \boldsymbol{r}_1'', \boldsymbol{r}_2'', t) . \qquad (66.6)$$

Hence Eq. (66.3) takes the form

$$\frac{\partial f_a}{\partial t} + \boldsymbol{v}_1 \cdot \frac{\partial f_a}{\partial \boldsymbol{r}_1} - \frac{i}{\hbar} (2\pi)^{-3} \int d\boldsymbol{\gamma}\, d\boldsymbol{\eta} \left[U_a(\boldsymbol{r}_1 - \tfrac{1}{2}\hbar\boldsymbol{\gamma}) - U_a(\boldsymbol{r}_1 + \tfrac{1}{2}\hbar\boldsymbol{\gamma})\right]$$

$$\cdot \exp\left[i\boldsymbol{\gamma} \cdot (\boldsymbol{\eta} - \boldsymbol{p}_1)\right] f_a(\boldsymbol{r}_1, \boldsymbol{\eta}, t) = I_a(\boldsymbol{r}_1, \boldsymbol{p}_1, t) . \qquad (66.7)$$

The collision integral I_a is defined as follows

$$I_a = \frac{i}{\hbar} (2\pi)^{-3} \frac{(2\pi\hbar)^3}{V^2} \sum_b N_b \int d\boldsymbol{\gamma}\, d\boldsymbol{r}_2 \left[\Phi_{ab}(\boldsymbol{r}_1 - \boldsymbol{r}_2 - \tfrac{1}{2}\hbar\boldsymbol{\gamma}) - \Phi_{ab}(\boldsymbol{r}_1 - \boldsymbol{r}_2 + \tfrac{1}{2}\hbar\boldsymbol{\gamma})\right]$$

$$e^{-i\boldsymbol{\gamma}\cdot\boldsymbol{p}_1} g_{ab}(\boldsymbol{r}_1 + \tfrac{1}{2}\hbar\boldsymbol{\gamma}, \boldsymbol{r}_1 - \tfrac{1}{2}\hbar\boldsymbol{\gamma}, \boldsymbol{r}_2, \boldsymbol{r}_2, t) \qquad (66.8)_1$$

and the potential of the average field is given by

$$U_a(\boldsymbol{r}_1, t) = U_{a0} + \sum_b N_b \frac{1}{V} \int d\boldsymbol{r}_2\, \Phi_{ab}(\boldsymbol{r}_1 - \boldsymbol{r}_2)\, \rho_b(\boldsymbol{r}_2, \boldsymbol{r}_2, t) . \qquad (66.9)$$

In the case of a spatially homogeneous gas, the collision integral reduces to

$$I_a(\boldsymbol{p}_1, t) = \frac{i}{\hbar} \frac{1}{V^3} \sum_b N_b \int d\boldsymbol{r}_1'\, d\boldsymbol{r}_1''\, d\boldsymbol{r}_2 \left[\Phi_{ab}(\boldsymbol{r}_1'' - \boldsymbol{r}_2) - \Phi_{ab}(\boldsymbol{r}_1' - \boldsymbol{r}_2)\right]$$

$$\cdot \exp\left[-i(\boldsymbol{r}_1' - \boldsymbol{r}_1'') \cdot \boldsymbol{p}_1/\hbar\right] \rho_{ab}(\boldsymbol{r}_1', \boldsymbol{r}_1'', \boldsymbol{r}_2, \boldsymbol{r}_2, t) . \qquad (66.8)_2$$

We used here the fact that for spatially homogeneous gases we may use ρ_{ab} instead of g_{ab} in the collision integral. In the classical limit, and for a one-component gas, (66.7) reduces to (2.11) and $(66.8)_1$ to (2.12).

We now consider the binary collision approximation. We then obtain for ρ_{ab} an equation analogous to the classical one (2.20) for f_2:

$$i\hbar \frac{\partial \rho_{ab}}{\partial t} + \left[\frac{\hbar^2}{2m_a}\left(\nabla^2_{r'_1} - \nabla^2_{r''_1}\right) + \frac{\hbar^2}{2m_b}\left(\nabla^2_{r'_2} - \nabla^2_{r''_2}\right)\right] \rho_{ab}$$

$$- \left[\Phi_{ab}(r'_1 - r'_2) - \Phi_{ab}(r''_1 - r''_2)\right] \rho_{ab}$$

$$= i\hbar \frac{\partial}{\partial t} \frac{V^2}{(2\pi\hbar)^6} \int d\boldsymbol{p}_1\, d\boldsymbol{p}_2\, f_a(\boldsymbol{p}_1, t)\, f_b(\boldsymbol{p}_2, t)$$

$$\cdot \exp\left[i\hbar^{-1}(r'_1 - r''_1)\cdot \boldsymbol{p}_1 + i\hbar^{-1}(r'_2 - r''_2)\cdot \boldsymbol{p}_2\right]$$

$$\equiv i\hbar \frac{\partial}{\partial t} \rho^0_{ab}(r'_1, r'_2, r''_1, r''_2, t). \tag{66.10}$$

Here ρ^0_{ab} denotes the two-particle density matrix for uncorrelated particles.

We now express the two-particle density matrix in terms of the eigenfunctions of the two-particle Hamiltonian, i.e. the functions obeying the following equation

$$\left[-\frac{\hbar^2}{2m_a}\nabla^2_{r_1} - \frac{\hbar^2}{2m_b}\nabla^2_{r_2} + \Phi_{ab}(r_1 - r_2)\right] \psi_{\boldsymbol{p}_1 \boldsymbol{p}_2}(r_1, r_2)$$

$$= E_{\boldsymbol{p}_1 \boldsymbol{p}_2}\, \psi_{\boldsymbol{p}_1 \boldsymbol{p}_2}(r_1, r_2) \tag{66.11}$$

$$\int dr_1\, dr_2\, \psi^*_{\boldsymbol{p}''_1 \boldsymbol{p}''_2}(r_1, r_2)\, \psi_{\boldsymbol{p}'_1 \boldsymbol{p}'_2}(r_1, r_2) = \frac{(2\pi\hbar)^6}{V^2}\, \delta(\boldsymbol{p}'_1 - \boldsymbol{p}''_1)\, \delta(\boldsymbol{p}'_2 - \boldsymbol{p}''_2). \tag{66.12}$$

The corresponding time-dependent functions are:

$$\psi_{\boldsymbol{p}_1 \boldsymbol{p}_2}(r_1, r_2, t) = \exp\left(-i\hbar^{-1} E_{\boldsymbol{p}_1 \boldsymbol{p}_2} t\right) \psi_{\boldsymbol{p}_1 \boldsymbol{p}_2}(r_1, r_2). \tag{66.13}$$

In the free-motion approximation we have

$$\psi_{\boldsymbol{p}_1 \boldsymbol{p}_2}(r_1, r_2) = V^{-1} \exp\left[i\hbar^{-1}(r_1 \cdot \boldsymbol{p}_1 + r_2 \cdot \boldsymbol{p}_2)\right]. \tag{66.14}$$

The expression of the density matrix is :

$$\rho_{ab}(r'_1, r''_1, r'_2, r''_2, t) = \left[\frac{V}{(2\pi\hbar)^3}\right]^4 \int d\boldsymbol{p}'_1\, d\boldsymbol{p}''_1\, d\boldsymbol{p}'_2\, d\boldsymbol{p}''_2$$

$$\rho_{ab}(\boldsymbol{p}'_1, \boldsymbol{p}''_1, \boldsymbol{p}'_2, \boldsymbol{p}''_2, t)\, \psi_{\boldsymbol{p}'_1 \boldsymbol{p}'_2}(r'_1, r'_2)\, \psi_{\boldsymbol{p}''_1 \boldsymbol{p}''_2}(r''_1, r''_2). \tag{66.15}$$

Using this relation, we find the equation for the density matrix in the momentum representation :

$$\left[i\hbar \frac{\partial}{\partial t} - \left(E_{\boldsymbol{p}'_1 \boldsymbol{p}'_2} - E_{\boldsymbol{p}''_1 \boldsymbol{p}''_2}\right)\right] \rho_{ab}(\boldsymbol{p}'_1, \boldsymbol{p}''_1, \boldsymbol{p}'_2, \boldsymbol{p}''_2, t)$$

$$= i\hbar \frac{\partial}{\partial t} \rho^0_{ab}(\boldsymbol{p}'_1, \boldsymbol{p}''_1, \boldsymbol{p}'_2, \boldsymbol{p}''_2, t). \tag{66.16}$$

The solution of this equation, assuming complete weakening of the initial correlations, is analogous to (12.1) :

$$\rho_{ab} = \exp\left[-i\hbar^{-1}\left(E_{p_1' p_2'} - E_{p_1'' p_2''}\right)\tau\right] \rho^0_{ab}(p_1', p_1'', p_2', p_2'', t-\tau)$$
$$+\int_0^\tau d\tau' \exp\left[-i\hbar^{-1}\left(E_{p_1' p_2'} - E_{p_1'' p_2''}\right)\frac{\partial}{\partial t}\rho^0_{ab}(p_1', p_1'', p_2', p_2'', t-\tau')\right. \tag{66.17}$$

Using this result and the relation (66.15) we find the density matrix in the coordinate representation. We first introduce the transition matrix :

$$A(-\tau) = \left[\frac{V}{(2\pi\hbar)^3}\right]^4 \int dp_1'\, dp_1''\, dp_2'\, dp_2''\, \psi_{p_1' p_2'}(r_1', r_2')\, \psi^*_{p_1'' p_2''}(r_1'', r_2'')$$
$$\cdot\, \psi^*_{p_1' p_2'}(q_1', q_2', -\tau)\, \psi_{p_1'' p_2''}(q_1'', q_2'', -\tau). \tag{66.18}$$

Then

$$\rho_{ab}(r_1', r_1'', r_2', r_2'') = \int dq_1'\, dq_1''\, dq_2'\, dq_2'' \Big\{A(-\tau)\, \rho^0_{ab}(q_1', q_1'', q_2', q_2'', t-\tau)$$
$$+\int_0^\tau d\tau'\, A(-\tau')\, \frac{\partial}{\partial t}\, \rho^0_{ab}(q_1', q_1'', q_2', q_2'', t-\tau')\Big\}\, . \tag{66.19}$$

This expression is analogous to the classical Eq. (12.1).

We substitute this relation into $(66.8)_2$, thus expressing the collision integral in terms of the one-particle distribution functions. We recall that ρ^0_{ab} is defined by Eq. (66.10).

Equation $(66.8)_2$ can be simplified if we retain only the first-order terms in $\tau_0(\partial/\partial t)$, i.e., the first-order retardation correction. We then represent the collision integral as in section 12, as a sum of two terms

$$I_a = I_{a1} + I_{a2}\, . \tag{66.20}$$

The first term, which is of zeroth order in the retardation, is obtained from (66.19)

$$\rho^1_{ab} = \int dq_1'\, dq_1''\, dq_2'\, dq_2''\, A(-\infty)\, \rho^0_{ab}(q_1', q_1'', q_2', q_2'', t).$$

For $\tau\to\infty$ in (66.18) we obtain

$$\psi^*_{p_1' p_2'}(q_1', q_2', -\tau)\, \psi_{p_1'' p_2''}(q_1'', q_2'', -\tau)$$
$$\longrightarrow V^{-2} \exp\left[-i\hbar^{-1}(p_1'\cdot q_1' + p_2'\cdot q_2')\right] \exp\left[i\hbar^{-1}(p_1''\cdot q_1'' + p_2''\cdot q_2'')\right]\, .$$

Indeed, to zeroth order in the retardation, the correlations in the initial state disappear completely. As a result, after integration over q_1', q_1'', q_2', q_2'' and p_1', p_1'', p_2', p_2'' we get

$$\rho^1_{ab}(r_1', r_1'', r_2', r_2'', t)$$

$$= \frac{V^4}{(2\pi\hbar)^6} \int dp_1\, dp_2\, \psi_{p_1, p_2}(r_1', r_2')\, \psi^*_{p_1, p_2}(r_1'', r_2'')$$

$$\cdot f_a(p_1, t)\, f_b(p_2, t)\ . \tag{66.21}$$

From $(66.8)_2$, (66.21) we get the following expression of the first part of the collision integral :

$$I_{a1}(p_1, t) = \frac{i}{\hbar} \frac{V^2}{(2\pi\hbar)^3} \frac{1}{(2\pi)^3} \sum_b N_b \int d\gamma\, dp_1'\, dr_2\, dp_2$$

$$\left[\Phi_{ab}(r_1 - r_2 - \tfrac{1}{2}\hbar\gamma) - \Phi_{ab}(r_1 - r_2 + \tfrac{1}{2}\hbar\gamma)\right] e^{-i\gamma \cdot p_1}$$

$$\psi_{p_1' p_2}(r_1 + \tfrac{1}{2}\hbar\gamma, r_2)\, \psi^*_{p_1' p_2}(r_1 - \tfrac{1}{2}\hbar\gamma, r_2)\, f_a(p_1, t)\, f_b(p_2, t). \tag{66.22}$$

This expression is identical to the one obtained by Green [1]. In reference [13] it was shown that this expression can be transformed into the usual form of the Boltzmann collision integral with a quantum cross-section.

The retardation correction gives an additional term to the Boltzmann equation as a result of the nonideality. The expression of the density matrix contributing this term is

$$\rho^{(2)}_{ab} = -\frac{\partial}{\partial t}\int d\tau \int dq_1'\, dq_1''\, dq_2'\, dq_2''\, \tau \frac{d}{d\tau} A(-\tau)\ \rho^0_{ab}(q_1', q_1'', q_2', q_2'', t). \tag{66.23}$$

Substituting this expression into $(66.8)_2$ we find the expression of I_{a2}. As in the classical case, this term gives the modifications of the non-dissipative quantities of the gas.

67. MACROSCOPIC EQUATIONS AND THERMODYNAMIC FUNCTIONS OF THE NONIDEAL GAS

As in the classical case, the collision integral has the following properties

$$\sum_a n_a \int dp\, \phi_a(p)\, I_a(p, t) = 0 \quad \text{for} \quad \phi_a = 1, p\ .$$

These properties ensure conservation of the number of particles and of the momentum.

Let us discuss in more detail the energy conservation equation. From (66.7) follows the conservation of the energy, including the pair collisions. Calculations similar to those of section 12 lead to the following expression for the internal energy density

$$U = \sum_a n_a \int dp\, \frac{V}{(2\pi\hbar)^3} \frac{p^2}{2m_a} f_a$$

$$+ \frac{1}{2} \sum_a \sum_b n_a n_b \int d(r_1 - r_2)\, \Phi_{ab}\, \rho^1_{ab}(r_1, r_1, r_2, r_2)\ . \tag{67.1}$$

Here ρ^1_{ab} is the distribution function of the particle positions to zeroth order in the retardation. Its expression follows from (66.21)

$$\rho^1_{ab}(\boldsymbol{r}_1, \boldsymbol{r}_1, \boldsymbol{r}_2, \boldsymbol{r}_2, t)$$
$$= \frac{V^4}{(2\pi\hbar)^6} \int d\boldsymbol{p}_1\, d\boldsymbol{p}_2 \left|\psi_{\boldsymbol{p}_1 \boldsymbol{p}_2}(\boldsymbol{r}_1, \boldsymbol{r}_2)\right|^2 f_a(\boldsymbol{p}_1, t)\, f_b(\boldsymbol{p}_1, t)\, . \quad (67.2)$$

In the expression of the internal energy, the interactions contribute not only to the potential energy (second term), but also to the kinetic energy. As we know from section 14, the one-particle distribution function can be expanded in powers of the density. To first order we obtain from (66.7) , (66.10) :

$$f_a(\boldsymbol{p}_1, t) = C\left[f^0_a + \sum_b n_b \frac{1}{(2\pi\hbar)^3} \int d\boldsymbol{r}_1\, d\boldsymbol{r}_2\, g_{ab}(x_1, x_2)\right] \quad (67.3)$$

which is analogous to the classical expression (14.4). C is a normalization constant, f^0_a is the distribution function of the ideal gas, and g_{ab} is the correlation function. Using the value of C we obtain, from (67.3), to first order in the density

$$f_a(\boldsymbol{p}_1, t) = f^0_a + \sum_b n_b \frac{1}{V} \int d\boldsymbol{r}_1\, d\boldsymbol{r}_2 \left\{\frac{V}{(2\pi\hbar)^3} \int d\boldsymbol{p}_2\, g_{ab}(x_1, x_2, t)\right.$$
$$\left. - \frac{V^2}{(2\pi\hbar)^6} \int d\boldsymbol{p}_1\, d\boldsymbol{p}_2\, f^0_a(\boldsymbol{p}_1, t)\, g_{ab}(x_1, x_2, t)\right\} . \quad (67.4)$$

On the right-hand side we need the expression of g_{ab} to zeroth order in the density and in the retardation. The function $f_a(\boldsymbol{p}, t)$ differs from the ideal gas distribution f^0_a because of two types of correlations: the quantum correlations, and the classical correlations due to interactions. In equilibrium only the quantum correlations remain, and they vanish as $\hbar \to 0$.

We now derive the expression of f_a, correct to order $\hbar^2$. We start from Eq. (66.3) for the Wigner function. We solve it by successive approximations in $\hbar$, starting from the classical Maxwell-Boltzmann distribution f^{cl}_{ab} for a pair of particles. In the binary collision approximation we get

$$f^{\mathrm{QU}}_a = f^{\mathrm{cl}}_a + \sum_b n_b \frac{\hbar^2}{24 k_{\mathrm{B}} T} \frac{1}{V} \int d\boldsymbol{r}_1\, d\boldsymbol{r}_2 \left(\frac{\partial \Phi_{ab}}{\partial \boldsymbol{r}_1} \cdot \frac{\partial}{\partial \boldsymbol{p}_1}\right)^2 f^{\mathrm{cl}}_a\, g^{\mathrm{cl}}_{ab}(\boldsymbol{r}_1, \boldsymbol{r}_2) \quad (67.5)$$

which agrees with Green's result [1]. Here g^{cl}_{ab} is the Boltzmann pair correlation in the binary collision approximation.

From (67.5) we obtain the well-known expression (sect. 33 of ref. [14]) of the average kinetic energy in the local-equilibrium state

$$\frac{V}{(2\pi\hbar)^3} \int d\boldsymbol{p}\, \frac{p^2}{2m_a} f^{\mathrm{QU}}_a = \tfrac{3}{2} k_{\mathrm{B}} T + \frac{\hbar^2}{24\, m_a k_{\mathrm{B}} T} \sum_b n_b \frac{1}{V} \int d\boldsymbol{r}_1\, d\boldsymbol{r}_2 \frac{\partial^2 \Phi_{ab}}{\partial \boldsymbol{r}_1^2}\, g^{\mathrm{cl}}_{ab} .$$

Equation (67.1) can be represented as follows :

$$U = U_{\mathbf{id}} + U_{\mathbf{cor}} \tag{67.6}$$

where $U_{\mathbf{id}}$ is the energy density of the ideal gas. The quantity $U_{\mathbf{cor}}$ represents the potential energy and the correlational part of the kinetic energy. We consider the latter quantity in the local-equilibrium state. From (67.4) follows that the correction to the kinetic energy is

$$U_{\mathbf{cor}}^{\mathbf{kin}} = \frac{1}{2}\sum_a \sum_b n_a n_b \frac{V}{(2\pi\hbar)^6} \int dx_1\, dx_2 \left(\frac{p_1^2}{2m_a} + \frac{p_2^2}{2m_a} - 3\, k_{\mathbf{B}} T\right) f_{ab}^0 \tag{67.7}$$

where f_{ab}^0 is the quantum equilibrium distribution function.

The function

$$U_{\mathbf{cor}} = U_{\mathbf{cor}}^{\mathbf{kin}} + U_{\mathbf{pot}}$$

can be expressed through the simple distribution $\rho_{ab}(\boldsymbol{r}_1, \boldsymbol{r}_1, \boldsymbol{r}_2, \boldsymbol{r}_2)$. Indeed, from (67.1), (67.7):

$$U_{\mathbf{cor}} = \frac{1}{2} \frac{V}{(2\pi\hbar)^6} \sum_a \sum_b n_a n_b \int dx_1\, dx_2 (H_{ab} - 3\, k_{\mathbf{B}} T)\, f_{ab}^0 .$$

Setting $\Phi_{ab}(\lambda) = \lambda \Phi_{ab}$, we rewrite this expression as

$$U_{\mathbf{cor}} = \frac{1}{2} \frac{V}{(2\pi\hbar)^6} \sum_a \sum_b n_a n_b \int_0^1 d\lambda \frac{\partial}{\partial\lambda} \int dx_1\, dx_2\, H_{ab}(\lambda)\, f_{ab}^0 . \tag{67.8}$$

On the other hand,

$$\frac{\partial}{\partial\lambda} \frac{1}{(2\pi\hbar)^6} \int dx_1\, dx_2\, H_{ab}(\lambda)\, f_{ab}^0 = - \frac{\partial^2}{\partial\beta\,\partial\lambda} \ln \operatorname{Tr} \exp\left[-\beta \hat{H}_{ab}(\lambda)\right]$$

$$= \frac{\partial}{\partial\beta} \beta \frac{1}{(2\pi\hbar)^6} \int dx_1\, dx_2 \frac{\Phi_{ab}(\lambda)}{\lambda} f_{ab}^0 = \frac{\partial}{\partial\beta} \frac{1}{V^2} \beta \int d\boldsymbol{r}_1\, d\boldsymbol{r}_2 \frac{\Phi_{ab}(\lambda)}{\lambda} \rho_{ab}^0 . \tag{67.9}$$

Substituting this into (67.8) and using the expression of the potential energy from (67.1) we get

$$U_{\mathbf{cor}} = \frac{1}{2} \frac{1}{V} \frac{\partial}{\partial\beta} \beta \sum_a \sum_b n_a n_b \int_0^1 d\lambda \int d\boldsymbol{r}_1\, d\boldsymbol{r}_2 \frac{\Phi_{ab}(\lambda)}{\lambda} \rho_{ab}^0$$

$$= \frac{\partial}{\partial\beta} \beta \int_0^1 d\lambda \frac{1}{\lambda} U_{\mathbf{pot}}(\lambda, \beta) . \tag{67.10}$$

This expression agrees with the well-known result from the equilibrium theory of quantum systems (see for instance, appendix 7 of ref. [15]). Equation (67.10) actually defines the second virial coefficient in the expansion of the internal energy in powers of the density.

68. TWO FORMS OF THE QUANTUM COLLISION INTEGRAL

In the classical theory we used two forms of the collision integral of an ideal gas. One of them was the usual Boltzmann form (10.18), the other being the Bogolyubov form (10.5). The proof of their equivalence made use of Eq. (10.8)

which follows (to zeroth order in $\tau(\partial/\partial t)$) from Eq. (10.2) for the two-particle distribution f_2.

An analogous equation can be easily derived in the quantum case from the function f_{ab}. It follows from (66.10) by using the relation between f_{ab} and ρ_{ab}. After some transformations similar to those of section 10, we obtain the second form of the collision integral (for homogeneous systems):

$$I_a(\boldsymbol{p}_1, t) = \frac{1}{(2\pi\hbar)^3 V} \sum_b N_b \int d\boldsymbol{r}_1\, d\boldsymbol{r}_2\, d\boldsymbol{p}_2 \left(\boldsymbol{v}_1 \cdot \frac{\partial}{\partial \boldsymbol{r}_1} + \boldsymbol{v}_2 \cdot \frac{\partial}{\partial \boldsymbol{r}_2}\right) f_{ab}(x_1, x_2, t)$$

Instead of f_{ab} we may as well use the correlation function $g_{ab} = f_{ab} - f_a f_b$, because f_a, f_b do not depend on the positions.

For one-component gases (68.1) corresponds to the classical expression (10.9). To make the comparison, one must recall the relation (66.5) between classical and quantum distribution functions.

For an ideal gas the expression for the function f_{ab} follows from (66.21), by using the relation between f_{ab} and ρ_{ab} which is analogous to (66.1). Thus, in the binary collision model and to zeroth order in the retardation we have

$$\begin{aligned} f_{ab}&(\boldsymbol{r}_1, \boldsymbol{p}_1, \boldsymbol{r}_2, \boldsymbol{p}_2, t) \\ &= \frac{V^2}{(2\pi)^6} \int d\boldsymbol{\Upsilon}_1\, d\boldsymbol{p}_1'\, d\boldsymbol{\Upsilon}_2\, d\boldsymbol{p}_2' \exp\left(-i\boldsymbol{\Upsilon}_1 \cdot \boldsymbol{p}_1 - i\boldsymbol{\Upsilon}_2 \cdot \boldsymbol{p}_2\right) \\ &\quad \psi_{\boldsymbol{p}_1' \boldsymbol{p}_2'}\left(\boldsymbol{r}_1 + \tfrac{1}{2}\hbar\boldsymbol{\Upsilon}_1, \boldsymbol{r}_2 + \tfrac{1}{2}\hbar\boldsymbol{\Upsilon}_2\right) \\ &\quad \psi^*_{\boldsymbol{p}_1' \boldsymbol{p}_2'}\left(\boldsymbol{r}_1 - \tfrac{1}{2}\hbar\boldsymbol{\Upsilon}_1, \boldsymbol{r}_2 - \tfrac{1}{2}\hbar\boldsymbol{\Upsilon}_2\right) f_a(\boldsymbol{p}_1', t)\, f_b(\boldsymbol{p}_2', t). \end{aligned} \tag{68.2}$$

We see that the function f_{ab} depends on time only through the one-particle distributions f_a, f_b.

Substituting (68.2) into (68.1) and performing the integrations in spherical coordinates we obtain the quantum expression of the Boltzmann collision integral. It differs from (10.18) by the change of the classical cross-section (10.20) to the quantum cross-section [13].

69. THE WEAK-COUPLING APPROXIMATION

It was shown in section 13 that the kinetic equation of a nonideal gas in the weak-coupling approximation can be obtained either by expanding the Boltzmann collision integral, or by expanding the correlation function g_{ab} in powers of the interactions. Both methods can be applied, of course, also in the quantum case.

The equation for the correlation density matrix in the weak-coupling approximation follows from (66.10):

$$i\hbar \frac{\partial g_{ab}}{\partial t} + \left[\frac{\hbar^2}{2m_a}\left(\nabla^2_{\boldsymbol{r}'_1} - \nabla^2_{\boldsymbol{r}''_1}\right) + \frac{\hbar^2}{2m_b}\left(\nabla^2_{\boldsymbol{r}'_2} - \nabla^2_{\boldsymbol{r}''_2}\right)\right] g_{ab}$$

$$= \left\{\Phi_{ab}(\boldsymbol{r}'_1 - \boldsymbol{r}'_2) - \Phi_{ab}(\boldsymbol{r}''_1 - \boldsymbol{r}''_2)\right\} \rho_a(\boldsymbol{r}'_1, \boldsymbol{r}''_1, t)\, \rho_b(\boldsymbol{r}'_2, \boldsymbol{r}''_2, t). \quad (69.1)$$

It corresponds to the classical relation (13.1).

We substitute here the expression (66.15) of g_{ab} in terms of the eigenfunctions to zeroth order in the interactions [i.e. (66.14)]. As a result we obtain

$$\left[i\hbar \frac{\partial}{\partial t} - \left(E_{\boldsymbol{p}'_1 \boldsymbol{p}'_2} - E_{\boldsymbol{p}''_1 \boldsymbol{p}''_2}\right)\right] g_{ab}(\boldsymbol{p}'_1, \boldsymbol{p}'_2, \boldsymbol{p}''_1, \boldsymbol{p}''_2, t)$$

$$= A(\boldsymbol{p}'_1, \boldsymbol{p}'_2, \boldsymbol{p}''_1, \boldsymbol{p}''_2, t) \quad (69.2)$$

where

$$E_{\boldsymbol{p}_1 \boldsymbol{p}_2} = \frac{p_1^2}{2m_a} + \frac{p_2^2}{2m_b}.$$

We introduced here the notation

$$A = \frac{1}{V^2} \int d\boldsymbol{r}'_1\, d\boldsymbol{r}'_2\, d\boldsymbol{r}''_1\, d\boldsymbol{r}''_2 \left[\Phi_{ab}(\boldsymbol{r}'_1 - \boldsymbol{r}'_2) - \Phi_{ab}(\boldsymbol{r}''_1 - \boldsymbol{r}''_2)\right]$$

$$\exp\left[i\hbar^{-1}(\boldsymbol{r}''_1 \cdot \boldsymbol{p}''_1 + \boldsymbol{r}''_2 \cdot \boldsymbol{p}''_2)\right] \exp\left[-i\hbar^{-1}(\boldsymbol{r}'_1 \cdot \boldsymbol{p}'_1 + \boldsymbol{r}'_2 \cdot \boldsymbol{p}'_2)\right]$$

$$\rho_a(\boldsymbol{r}'_1, \boldsymbol{r}''_1, t)\, \rho_b(\boldsymbol{r}''_1, \boldsymbol{r}''_2, t). \quad (69.3)$$

For spatially homogeneous distributions

$$\rho_a(\boldsymbol{r}'_1, \boldsymbol{r}''_1, t) = \rho_a(\boldsymbol{r}'_1 - \boldsymbol{r}''_1, t)$$

$$= \frac{V}{(2\pi\hbar)^3} \int d\boldsymbol{p}_1 \exp\left[i\hbar^{-1}(\boldsymbol{r}'_1 - \boldsymbol{r}''_1) \cdot \boldsymbol{p}_1\right] f_a(\boldsymbol{p}_1, t). \quad (69.4)$$

We use this expression, use the Fourier representation of the potential $\Phi_{ab}(\boldsymbol{r})$ in (69.3) and integrate over the positions, with the result :

$$A = (2\pi\hbar)^3\, \nu_{ab} \frac{|\boldsymbol{p}'_1 - \boldsymbol{p}''_1|}{\hbar}\, \delta(\boldsymbol{p}'_1 + \boldsymbol{p}'_2 - \boldsymbol{p}''_1 - \boldsymbol{p}''_2)$$

$$\cdot \left[f_a(\boldsymbol{p}''_1, t)\, f_b(\boldsymbol{p}''_2, t) - f_a(\boldsymbol{p}'_1, t)\, f_b(\boldsymbol{p}'_2, t)\right] \quad (69.5)$$

where

$$\nu_{ab}(\boldsymbol{k}) = \int d\boldsymbol{r}\, e^{-i\boldsymbol{k}\cdot\boldsymbol{r}}\, \Phi_{ab}(\boldsymbol{r}). \quad (69.6)$$

We rewrite Eq. (69.2) by assuming complete weakening of the initial correlations :

$$g_{ab}(\boldsymbol{p}'_1, \boldsymbol{p}'_2, \boldsymbol{p}''_1, \boldsymbol{p}''_2, t)$$

$$= -i \int_0^\infty d\tau\, \hbar^{-1} \exp\left[-i\hbar^{-1}\left(E_{\boldsymbol{p}'_1 \boldsymbol{p}'_2} - E_{\boldsymbol{p}''_1 \boldsymbol{p}''_2}\right)\tau\right]$$

$$A(\boldsymbol{p}'_1, \boldsymbol{p}'_2, \boldsymbol{p}''_1, \boldsymbol{p}''_2, t-\tau). \quad (69.7)$$

We now transform the expression $(66.8)_2$ for the collision integral. We substitute the expression (66.15) for g_{ab} in terms of the eigenfunctions (66.14), Fourier-transform the potential, and perform all possible integrations :

$$I_a(\boldsymbol{p}_1, t) = \frac{2}{\hbar} \frac{1}{(2\pi\hbar)^6} \sum_b \frac{N_b}{V} \int d\boldsymbol{p}_1' \, d\boldsymbol{p}_2' \, d\boldsymbol{p}_2 \; \nu_{ab}(\hbar^{-1}|\boldsymbol{p}_1 - \boldsymbol{p}_1'|)$$

$$\delta(\boldsymbol{p}_1 + \boldsymbol{p}_2 - \boldsymbol{p}_1' - \boldsymbol{p}_2') \; \mathrm{Im}\, g_{ab}(\boldsymbol{p}_1', \boldsymbol{p}_2', \boldsymbol{p}_1, \boldsymbol{p}_2, t) \, . \tag{69.8}$$

From (69.8), (69.7) and (69.5) we obtain the desired expression of the collision integral for a weakly coupled nonideal gas.

$$I_a(\boldsymbol{p}_1, t) = \frac{2}{\hbar} \frac{1}{(2\pi\hbar)^6} \sum_b N_b \int_0^\infty d\tau \int d\boldsymbol{p}_1' \, d\boldsymbol{p}_2' \, d\boldsymbol{p}_2 \; \nu_{ab}^2(\hbar^{-1}|\boldsymbol{p}_1 - \boldsymbol{p}_1'|)$$

$$\delta(\boldsymbol{p}_1 + \boldsymbol{p}_2 - \boldsymbol{p}_1' - \boldsymbol{p}_2') \cos\left[\hbar^{-1}\left(E_{\boldsymbol{p}_1 \boldsymbol{p}_2} - E_{\boldsymbol{p}_1' \boldsymbol{p}_2'}\right)\tau\right]$$

$$\left[f_a(\boldsymbol{p}_1', t-\tau) \, f_b(\boldsymbol{p}_2', t-\tau) - f_a(\boldsymbol{p}_1, t-\tau) \, f_b(\boldsymbol{p}_2, t-\tau)\right] . \tag{69.9}$$

To first order in the retardation, this expression can again be represented as a sum of two terms

$$I_a(\boldsymbol{p}_1, t) = I_{a1} + I_{a2} \tag{69.10}$$

where

$$I_{a1} = \frac{2\pi}{\hbar} \frac{1}{(2\pi\hbar)^6} \sum_b N_b \int d\boldsymbol{p}_1' \, d\boldsymbol{p}_2' \, d\boldsymbol{p}_2 \; \nu_{ab}^2(\hbar|\boldsymbol{p}_1 - \boldsymbol{p}_1'|)$$

$$\delta\left[\frac{p_1^2}{2m_a} + \frac{p_2^2}{2m_b} - \frac{p_1'^2}{2m_a} - \frac{p_2'^2}{2m_b}\right] \delta(\boldsymbol{p}_1 + \boldsymbol{p}_2 - \boldsymbol{p}_1' - \boldsymbol{p}_2')$$

$$\left[f_a(\boldsymbol{p}_1', t) \, f_b(\boldsymbol{p}_2', t) - f_a(\boldsymbol{p}_1, t) \, f_b(\boldsymbol{p}_2, t)\right] \tag{69.11}$$

is the collision integral of a weakly coupled ideal gas. This expression was first obtained by Bogolyubov and Gurov [11]†. The integral I_{a2} is:

$$I_{a2} = -\frac{\partial}{\partial t} \frac{2}{\hbar^2} \cdot \frac{1}{(2\pi\hbar)^6} \sum_b N_b \int d\tau \; \tau \int d\boldsymbol{p}_1' \, d\boldsymbol{p}_2' \, d\boldsymbol{p}_2 \nu_{ab}^2(\hbar^{-1}|\boldsymbol{p}_1 - \boldsymbol{p}_1'|)$$

$$\delta(\boldsymbol{p}_1 + \boldsymbol{p}_2 - \boldsymbol{p}_1' - \boldsymbol{p}_2') \cos\left[\hbar^{-1}\left(E_{\boldsymbol{p}_1 \boldsymbol{p}_2} - E_{\boldsymbol{p}_1' \boldsymbol{p}_2'}\right)\tau\right]$$

$$\Big[f_a(\boldsymbol{p}_1', t) \; f_b(\boldsymbol{p}_2', t) - f_a(\boldsymbol{p}_1, t) \, f_b(\boldsymbol{p}_1', t)$$

This expression modifies the non-dissipative properties of the gas, i.e., takes account to the nonideality.

† **This is actually a particular case of the kinetic equation derived by Uehling and Uhlenbeck in 1933 [49], which is the full quantum generalization of the Boltzmann equation, taking also quantum statistics into account; see also [47]. (Transl.)**

Here ρ^0_{ab} is the two-particle density matrix for an uncorrelated system. The right-hand side of (70.3) differs from (66.10) because it takes into account the exchange effects. The upper sign (+) corresponds to the case of Bose statistics, and the lower sign (−) corresponds to Fermi statistics.

We shall not consider here processes which involve the transition of the particles from one spin state to another. Therefore, the functions (70.1) depend on the spin variables as follows :

$$f_a(\sigma_1', \sigma_1'', \boldsymbol{p}_1, t) = \frac{1}{g_a} \delta_{\sigma_1', \sigma_1''} f_a(\boldsymbol{p}_1, t)$$

$$\rho_{ab}(\sigma_1', \sigma_1'', \sigma_2', \sigma_2'', \boldsymbol{r}_1', \boldsymbol{r}_1'', \boldsymbol{r}_2', \boldsymbol{r}_2'', t)$$

$$= \frac{1}{g_a g_b} \delta_{\sigma_1', \sigma_1''} \delta_{\sigma_2', \sigma_2''} \rho_{ab}(\boldsymbol{r}_1', \boldsymbol{r}_1'', \boldsymbol{r}_2', \boldsymbol{r}_2'', t) \tag{70.4}$$

where $g_a = 2S_a + 1$, S_a being the value of the spin in $\hbar$ as unit.

The functions (70.1), (70.4) are normalized as follows

$$\sum_\sigma \frac{V}{(2\pi\hbar)^3} \int d\boldsymbol{p}\, f_a(\sigma, \sigma, \boldsymbol{p}, t) = 1$$

$$\sum_{\sigma_1} \sum_{\sigma_2} \frac{1}{V^2} \int d\boldsymbol{r}_1\, d\boldsymbol{r}_2\, \rho_{ab}(\sigma_1, \sigma_1, \sigma_2, \sigma_2, \boldsymbol{r}_1, \boldsymbol{r}_1, \boldsymbol{r}_2, \boldsymbol{r}_2, t) = 1. \tag{70.5}$$

The subsequent calculations proceed as in section 66. The result is analogous to (66.19) and can be represented in the form

$$\rho_{ab} = \rho^1_{ab} + \rho^{(2)}_{ab} \tag{70.6}$$

where

$$\rho^1_{ab} = \frac{V^4}{(2\pi\hbar)^6} \int d\boldsymbol{p}_1\, d\boldsymbol{p}_2 \Big\{ \psi_{\boldsymbol{p}_1 \boldsymbol{p}_2}(\boldsymbol{r}_1', \boldsymbol{r}_2')\, \psi^*_{\boldsymbol{p}_1 \boldsymbol{p}_2}(\boldsymbol{r}_1'', \boldsymbol{r}_2'')$$

$$f_a(\sigma_1', \sigma_1'', \boldsymbol{p}_1, t)\, f_b(\sigma_2', \sigma_2'', \boldsymbol{p}_2, t)$$

$$\pm \delta_{ab}\, \psi_{\boldsymbol{p}_1 \boldsymbol{p}_2}(\boldsymbol{r}_1', \boldsymbol{r}_2'')\, \psi^*_{\boldsymbol{p}_1 \boldsymbol{p}_2}(\boldsymbol{r}_2', \boldsymbol{r}_1'')$$

$$f_a(\sigma_1', \sigma_2'', \boldsymbol{p}_1, t)\, f_b(\sigma_2', \sigma_1'', \boldsymbol{p}_2, t) \Big\} \tag{70.7}$$

$$\rho^{(2)}_{ab} = -\frac{\partial}{\partial t} \int_0^\infty d\tau \int d\boldsymbol{q}_1'\, d\boldsymbol{q}_1''\, d\boldsymbol{q}_2'\, d\boldsymbol{q}_2''\, \tau \frac{d}{d\tau} A(-\tau)$$

$$\cdot\, \rho^0_{ab}(\sigma_1', \sigma_1'', \sigma_2', \sigma_2'', \boldsymbol{q}_1', \boldsymbol{q}_1'', \boldsymbol{q}_2', \boldsymbol{q}_2'', t) \tag{70.8}$$

where $A(-\tau)$ is the transition matrix, defined as follows :

$$A(-\tau) = \frac{V^4}{(2\pi\hbar)^{12}} \int dp_1' \, dp_2' \, dp_1'' \, dp_2'' \; \psi_{p_1' p_2'}(r_1', r_2') \, \psi^*_{p_1'' p_2''}(r_1'', r_2'')$$

$$\psi^*_{p_1' p_2'}(q_1', q_2', -\tau) \, \psi_{p_1'' p_2''}(q_1'', q_2'', -\tau) \,. \tag{70.9}$$

The collision integral, to first order in the retardation, is represented as

$$I_a(\sigma_1', \sigma_1'', p_1, t) = I_{a1} + I_{a2} \tag{70.10}$$

From (70.2), (70.7) we obtain:

$$I_{a1} = \frac{i}{\hbar} \frac{V^2}{(2\pi)^6 \hbar^3} \sum_b \sum_{\sigma_2} N_b \int d\gamma \, dp_1' \, dp_2 \, dr_2 \; e^{-i\gamma \cdot p_1}$$

$$\left[\Phi_{ab}(r_1 - r_2 - \tfrac{1}{2}\hbar\gamma) - \Phi_{ab}(r_1 - r_2 + \tfrac{1}{2}\hbar\gamma)\right]$$

$$\Big\{ \psi_{p_1' p_2}(r_1 + \tfrac{1}{2}\hbar\gamma, r_2) \, \psi^*_{p_1' p_2}(r_1 - \tfrac{1}{2}\hbar\gamma, r_2) \, f_a(\sigma_1', \sigma_1'', p_1, t) \, f_b(\sigma_2, \sigma_2, p_2, t)$$

$$\pm \delta_{ab} \, \psi_{p_1' p_2}(r_1 + \tfrac{1}{2}\hbar\gamma, r_2) \, \psi^*_{p_1' p_2}(r_2, r_1 - \tfrac{1}{2}\hbar\gamma) f_a(\sigma_1', \sigma_2, p_1', t) \, f_b(\sigma_2, \sigma_1'', p_2, t) \Big\}. \tag{70.11}$$

This is the quantum Boltzmann collision integral, taking account of the exchange effects, for an ideal non-degenerate gas.

The term I_{a2} is obtained from (70.8) and (70.2): it provides the effect of the interaction and exchange correlations on the non-dissipative properties of the gas.

If the spin dependence is given by (70.4), Eq. (70.2) reduces to the simpler equation

$$\frac{\partial f_a(p_1, t)}{\partial t} = \sum_{\sigma_1} I_a(\sigma_1, \sigma_1, p_1, t) \equiv I_a(p_1, t) \,. \tag{70.12}$$

In order to obtain the expression for I_{a1}, we substitute (70.4) into (70.11), perform the summations over σ_1 and use the identities

$$\sum_{\sigma_1} \sum_{\sigma_2} \frac{1}{g_a g_b} \delta_{\sigma_1' \sigma_2} \delta_{\sigma_2 \sigma_1''} = \sum_{\sigma_1} \frac{1}{g_a g_b} \delta_{\sigma_1' \sigma_1''} = \frac{1}{g_b}$$

We then obtain the result:

$$I_{a1} = \frac{i}{\hbar}\frac{V^2}{(2\pi)^6\hbar^3}\sum_b N_b \int d\boldsymbol{\Upsilon}\, d\boldsymbol{p}_1'\, d\boldsymbol{p}_2\, d\boldsymbol{r}_2\, e^{-i\boldsymbol{\Upsilon}\cdot\boldsymbol{p}_1}$$

$$\left[\Phi_{ab}(\boldsymbol{r}_1-\boldsymbol{r}_2-\tfrac{1}{2}\hbar\boldsymbol{\Upsilon}) - \Phi_{ab}(\boldsymbol{r}_1-\boldsymbol{r}_2+\tfrac{1}{2}\hbar\boldsymbol{\Upsilon})\right]$$

$$\Big\{\psi_{\boldsymbol{p}_1'\boldsymbol{p}_2}(\boldsymbol{r}_1+\tfrac{1}{2}\hbar\boldsymbol{\Upsilon},\boldsymbol{r}_2)\,\psi^*_{\boldsymbol{p}_1'\boldsymbol{p}_2}(\boldsymbol{r}_1-\tfrac{1}{2}\hbar\boldsymbol{\Upsilon},\boldsymbol{r}_2)$$

$$\pm\,\delta_{ab}\frac{1}{g_a}\psi_{\boldsymbol{p}_1'\boldsymbol{p}_2}(\boldsymbol{r}_1+\tfrac{1}{2}\hbar\boldsymbol{\Upsilon},\boldsymbol{r}_2)\,\psi^*_{\boldsymbol{p}_1'\boldsymbol{p}_2}(\boldsymbol{r}_2,\boldsymbol{r}_1-\tfrac{1}{2}\hbar\boldsymbol{\Upsilon})\Big\}$$

$$f_a(\boldsymbol{p}_1',t)\,f_b(\boldsymbol{p}_2,t)\,. \tag{70.13}$$

In the absence of exchange effects, this reduces to (66.22). In the weak coupling approximation, it reduces to the simpler form [16]:

$$I_{a1} = \frac{2\pi}{\hbar}\frac{1}{(2\pi\hbar)^6}\sum_b N_b\int d\boldsymbol{p}_1'\,d\boldsymbol{p}_2'\,d\boldsymbol{p}_2\,\delta(\boldsymbol{p}_1+\boldsymbol{p}_2-\boldsymbol{p}_1'-\boldsymbol{p}_2')$$

$$\delta\left(\frac{p_1^2}{2m_a}+\frac{p_2^2}{2m_b}-\frac{p_1'^2}{2m_a}-\frac{p_2'^2}{2m_b}\right)\Big[\nu_{ab}^2(\hbar^{-1}|\boldsymbol{p}_1-\boldsymbol{p}_1'|)$$

$$\pm\;\delta_{ab}\frac{1}{g_a}\nu_{ab}(\hbar^{-1}|\boldsymbol{p}_1-\boldsymbol{p}_2'|)\,\nu_{ab}(\hbar^{-1}|\boldsymbol{p}_1-\boldsymbol{p}_1'|)\Big]$$

$$\Big\{f_a(\boldsymbol{p}_1',t)\,f_b(\boldsymbol{p}_2',t)-f_a(\boldsymbol{p}_1,t)\,f_b(\boldsymbol{p}_2,t)\Big\} \tag{70.14}$$

which reduces to (69.11) in absence of exchange effects.

For a one-component gas of spinless particles the kinetic equation with this collision integral agrees with Eq. (10.41) of Gurov's book [11].

We now proceed to the study of the quantum kinetic theory of nonideal plasmas. In the calculation of the thermodynamic functions we shall stress the important rôle of the exchange effects in these systems.

CHAPTER 13

Quantum Kinetic Equations for Plasmas

71. THE PHASE-SPACE DENSITY OPERATOR

In sections 4 and 5 it was shown that the equation for the six-dimensional phase space density (4.6), (4.7) can be taken as a convenient starting point instead of the Liouville equation. The distribution functions $f_1, f_2, \ldots$ are related to the moments of this random function. We saw that this formalism is particularly effective in the theory of long-range fluctuations in gases and in the kinetic theory of plasmas.

In quantum theory, instead of the phase density

$$N_a(x,t) = \sum_{i=1}^{N_a} \delta(x-x_i(t)) \tag{71.1}$$

where $x = (r,p)$, we use the phase-density operator for spinless particles

$$\hat{N}_a(x,t) = \frac{1}{2\pi^3 V} (2\pi\hbar)^3 \int d\boldsymbol{\Upsilon}\, e^{-i\boldsymbol{\Upsilon}\cdot\boldsymbol{p}}\, \hat{\rho}_a(\boldsymbol{r}-\tfrac{1}{2}\hbar\boldsymbol{\Upsilon},\, \boldsymbol{r}-\tfrac{1}{2}\hbar\boldsymbol{\Upsilon}) \tag{71.2}$$

where

$$\frac{1}{V}\hat{\rho}_a(\boldsymbol{r}', \boldsymbol{r}'', t) = \psi_a^+(\boldsymbol{r}'', t)\, \psi_a(\boldsymbol{r}', t) \tag{71.3}$$

is the density matrix, ψ_a^+, ψ_a are creation and destruction operators, satisfying the following commutation relations [17] :

$$\psi_a(\boldsymbol{r},t)\, \psi_b^+(\boldsymbol{r}',t) \pm \psi_b^+(\boldsymbol{r}',t)\, \psi_a(\boldsymbol{r},t) = \delta_{ab}\, \delta(\boldsymbol{r}-\boldsymbol{r}')$$

$$\psi_a(\boldsymbol{r},t)\, \psi_b(\boldsymbol{r}',t) \pm \psi_b(\boldsymbol{r}',t)\, \psi_a(\boldsymbol{r},t) = 0$$

$$\psi_a^+(\boldsymbol{r},t)\, \psi_b^+(\boldsymbol{r}',t) \pm \psi_b^+(\boldsymbol{r}',t)\, \psi_a^+(\boldsymbol{r},t) = 0$$

The + sign corresponds to Fermi statistics, and the − sign to Bose statistics. The Hamiltonian for spinless particles is [17]:

$$\hat{H} = -\sum_a \frac{\hbar^2}{2m_a} \int d\boldsymbol{r}\, \psi_a^+(\boldsymbol{r},t)\, \nabla_{\boldsymbol{r}}^2\, \psi_a(\boldsymbol{r},t)$$

$$+ \frac{1}{2} \sum_a \sum_b \int d\boldsymbol{r}_1\, d\boldsymbol{r}_2\, \Phi_{ab}(\boldsymbol{r}_1 - \boldsymbol{r}_2)\, \psi_a^+(\boldsymbol{r}_1,t)\, \psi_b^+(\boldsymbol{r}_2,t)$$

$$\psi_b(\boldsymbol{r}_2,t)\, \psi_a(\boldsymbol{r}_1,t)\,. \tag{71.4}$$

We express this Hamiltonian in terms of the phase-density operators by using (71.2), (71.3). If the pairing effects due to exchange are neglected, we obtain

$$\hat{H} = \sum_a \frac{1}{(2\pi\hbar)^3} \int dx\, \frac{p^2}{2m_a}\, \hat{N}_a(x,t)$$

$$+ \frac{1}{2} \frac{1}{(2\pi\hbar)^6} \sum_a \sum_b \int dx\, dx'\, \Phi_{ab}(\boldsymbol{r} - \boldsymbol{r}')\, \hat{N}_a(x,t)\, \hat{N}_b(x',t)\,. \tag{71.5}$$

The second term can be symmetrized in $a, x\ ;\ b, x'$.

Equation (71.5) corresponds to the classical equation (4.3); it also retains the self-interaction term.

The equation for the phase-density operator can be obtained by means of the well known equations of motion for the operators ψ^+, ψ, using (71.2). In absence of an external field, we find [18] :

$$\left(\frac{\partial}{\partial t} + \boldsymbol{v}\cdot\frac{\partial}{\partial \boldsymbol{r}}\right) \hat{N}_a(x,t) - \frac{i}{\hbar} \frac{1}{(2\pi)^3} \frac{1}{(2\pi\hbar)^3} \sum_b \int d\boldsymbol{\gamma}\, d\boldsymbol{\eta}\, d\boldsymbol{r}'\, d\boldsymbol{p}'\, e^{i\boldsymbol{\gamma}\cdot(\boldsymbol{\eta}-\boldsymbol{p})}$$

$$\left[\Phi_{ab}(\boldsymbol{r}-\boldsymbol{r}'-\tfrac{1}{2}\hbar\boldsymbol{\gamma}) - \Phi_{ab}(\boldsymbol{r}-\boldsymbol{r}'+\tfrac{1}{2}\hbar\boldsymbol{\gamma})\right] \hat{N}_b(x',t)\, \hat{N}_a(\boldsymbol{r},\boldsymbol{\eta},t)\,. \tag{71.6}$$

This equation corresponds to the classical equation (4.7). We now introduce the potential energy operator

$$\hat{U}_a(\boldsymbol{r},t) = \frac{1}{(2\pi\hbar)^3} \sum_b \int dx'\, \Phi_{ab}(\boldsymbol{r} - \boldsymbol{r}')\, \hat{N}_b(x',t)\,. \tag{71.7}$$

We may then write (71.6) in a different form [18] :

$$\left(\frac{\partial}{\partial t} + \boldsymbol{v}\cdot\frac{\partial}{\partial \boldsymbol{r}}\right) \hat{N}_a(x,t) - \frac{i}{\hbar} \frac{1}{(2\pi)^3} \int d\boldsymbol{\eta}\, d\boldsymbol{\gamma}\, e^{i\boldsymbol{\gamma}\cdot(\boldsymbol{\eta}-\boldsymbol{p})}$$

$$\left[U_a(\boldsymbol{r}-\tfrac{1}{2}\hbar\boldsymbol{\gamma}) - U_a(\boldsymbol{r}+\tfrac{1}{2}\hbar\boldsymbol{\gamma})\right] \hat{N}_a(\boldsymbol{r},\boldsymbol{\eta},t)\,. \tag{71.8}$$

This equation corresponds to the classical equation (4.6). Using (71.7) we rewrite the Hamiltonian as follows

$$\hat{H} = \frac{1}{(2\pi\hbar)^3} \sum_a \int dx \left(\frac{p^2}{2m_a} + U_{a0}\right) \hat{N}_a(x,t)$$

$$+ \frac{1}{2} \frac{1}{(2\pi\hbar)^3} \sum_a \int dx\, U_a(\boldsymbol{r},t)\, \hat{N}_a(x,t)\,. \tag{71.9}$$

This expression can also be used for the description of exchange processes, if instead of (71.7) we use the following expression :

$$\hat{U}_a(x,t) = \frac{1}{(2\pi\hbar)^3}\sum_b \int dx' \Big[\Phi_{ab}(\boldsymbol{r}-\boldsymbol{r}') \mp \delta_{ab}\,\nu_{aa}(\hbar^{-1}|\boldsymbol{p}-\boldsymbol{p}'|)\,\delta(\boldsymbol{r}-\boldsymbol{r}')\Big]\,\hat{N}_a(x',t)\,. \tag{71.10}$$

The upper (lower) sign corresponds to Fermi (Bose) statistics.

The expressions (71.9), (71.10) correspond to (36) in ref. [18]. In order to derive (71.10) from (71.4), we must consider all possible pairings of the form $\psi^+\psi$.

If we consider particles with spin, but do not retain transitions between different spin states, we must use the expression :

$$\hat{U}_a(x,t) = \frac{1}{(2\pi\hbar)^3}\sum_b \int dx' \Big[\Phi_{ab}(\boldsymbol{r}-\boldsymbol{r}') \mp \frac{1}{g_a}\,\delta_{ab}\,\nu_{aa}(\hbar^{-1}|\boldsymbol{p}-\boldsymbol{p}'|)\,\delta(\boldsymbol{r}-\boldsymbol{r}')\Big]\cdot\hat{N}_b(x',t) \tag{71.11}$$

where $g_a = 2S_a + 1$. Thus, when the exchange effects are retained, the potential is replaced by

$$\Phi_{ab}(\boldsymbol{r}-\boldsymbol{r}') \mp \delta_{ab}\,\frac{1}{g_a}\,\nu_{aa}(\hbar^{-1}|\boldsymbol{p}-\boldsymbol{p}'|)\,\delta(\boldsymbol{r}-\boldsymbol{r}') \tag{71.12}$$

If the exchange effects are taken into account, (71.8) must be replaced by a more general equation, corresponding to the Hamiltonian (71.9) – (71.11). The added complication is due to the fact that the corresponding potential energy operator depends on the momenta as well as the positions. We thus obtain

$$\left(\frac{\partial}{\partial t} + \boldsymbol{v}\cdot\frac{\partial}{\partial \boldsymbol{r}}\right)\hat{N}_a(x,t) - \frac{i}{\hbar}\,\frac{1}{(2\pi)^6}\int d\boldsymbol{\gamma}\,d\boldsymbol{\eta}\,d\boldsymbol{\theta}\,d\boldsymbol{q}\,\exp\Big[i\boldsymbol{\gamma}\cdot(\boldsymbol{\eta}-\boldsymbol{p}) + i\boldsymbol{\theta}\cdot(\boldsymbol{q}-\boldsymbol{r})\Big]$$
$$\Big[\hat{U}_a(\boldsymbol{r}-\tfrac{1}{2}\hbar\boldsymbol{\gamma},\boldsymbol{p}+\tfrac{1}{2}\hbar\boldsymbol{\theta}) - \hat{U}_a(\boldsymbol{r}+\tfrac{1}{2}\hbar\boldsymbol{\gamma},\boldsymbol{p}-\tfrac{1}{2}\hbar\boldsymbol{\theta})\Big]\cdot\hat{N}_a(\boldsymbol{q},\boldsymbol{\eta},t) = 0\,. \tag{71.13}$$

For the case of fermions, we may use a simpler equation, if the momenta of the particles differ but slightly from the Fermi momentum. We may then expand (71.13) in powers of $\hbar\boldsymbol{\gamma}, \hbar\boldsymbol{\theta}$ and obtain

$$\left\{\frac{\partial}{\partial t} + \left(\boldsymbol{v} + \frac{\partial \hat{U}_a}{\partial \boldsymbol{p}}\right)\cdot\frac{\partial}{\partial \boldsymbol{r}} - \frac{\partial \hat{U}_a}{\partial \boldsymbol{r}}\cdot\frac{\partial}{\partial \boldsymbol{p}}\right\}\hat{N}_a(x,t) = 0\,, \tag{71.14}$$

where $\hat{U}_a$ is given by (71.11).

In the case of short-range forces the potential Φ_{ab} in (71.11) can be approximated by :

$$\Phi_{ab}(\boldsymbol{r}-\boldsymbol{r}') = \nu(0)\ \delta(\boldsymbol{r}-\boldsymbol{r}') . \tag{71.15}$$

This approximation is currently used in the theory of Fermi liquids [19,20].

72. THE EQUATIONS FOR THE MOMENTS

We go over from the operator equations (71.6) , (71.8) to the hierarchy of equations for the moments. We denote by $\langle\ \rangle$ brackets the average over the occupation numbers [17] :

$$\langle \hat{N}_a(x,t) \rangle = N_a f_a(x,t) \tag{72.1}$$

where f_a is the one-particle quantum distribution function (66.1).

For the two-body moment we obtain the following equation, not taking into account the exchange effects (we suppress now the caret ^ over N) :

$$\begin{aligned} \langle N_a(x,t)\, N_b(x',t)\rangle &= (N_a N_b - \delta_{ab} N_a)\, f_{ab}(x,x',t) \\ &+ \delta_{ab} N_a \frac{(2\pi\hbar)^3}{(2\pi)^6} \int d\boldsymbol{\gamma}\, d\boldsymbol{\eta}\ d\boldsymbol{\gamma}'\ d\boldsymbol{\eta}'\ \delta\left[\boldsymbol{r}-\boldsymbol{r}'+\tfrac{1}{2}\hbar(\boldsymbol{\gamma}+\boldsymbol{\gamma}')\right] \\ &\delta(\boldsymbol{\eta}-\boldsymbol{\eta}')\exp\left[i\boldsymbol{\gamma}\cdot(\boldsymbol{\eta}-\boldsymbol{p}) + i\boldsymbol{\gamma}'\cdot(\boldsymbol{\eta}'-\boldsymbol{p}')\right] \\ &f_a\left[\tfrac{1}{2}(\boldsymbol{r}+\boldsymbol{r}'-\tfrac{1}{2}\hbar\boldsymbol{\gamma}+\tfrac{1}{2}\hbar\boldsymbol{\gamma}'),\boldsymbol{\eta},t\right] , \end{aligned} \tag{72.2}$$

where f_{ab} is the two-particle quantum distribution function. In the classical limit this equation reduces to

$$\begin{aligned} \langle N_a(x,t)\, N_b(x',t)\rangle &= (N_a N_b - \delta_{ab} N_a)\, f_{ab} \\ &+ N_a\,\delta_{ab}(2\pi\hbar)^3\ \delta(x-x')\ f_a(x,t) . \end{aligned} \tag{72.3}$$

This equation agrees, for one-component systems, with the classical equation (5.2) (recalling (66.5)). For spatially homogeneous systems (72.2) simplifies :

$$\begin{aligned} \langle N_a(x,t)\ N_b(x',t)\rangle &= (N_a N_b - \delta_{ab} N_a)\ f_{ab} \\ &+ N_a\,\delta_{ab}(2\pi\hbar)^3\ \delta(\boldsymbol{p}-\boldsymbol{p}')\int d\boldsymbol{\gamma}\, d\boldsymbol{\eta}\ \delta(\boldsymbol{r}-\boldsymbol{r}'+\tfrac{1}{2}\hbar\boldsymbol{\gamma})\ e^{i\boldsymbol{\gamma}\cdot(\boldsymbol{\eta}-\boldsymbol{p})}\ f_a(\boldsymbol{\eta},t) . \end{aligned} \tag{72.4}$$

From (72.2) and (72.5) we derive the relation between the second central moment and the quantum correlation function. For instance, for spatially homogeneous systems we find

$$\begin{aligned} \langle \delta N_a\ \delta N_b\rangle_{x,x',t} &= (N_a N_b - \delta_{ab} N_a)\ g_{ab} \\ &+ N_a\,\delta_{ab}\Big\{\delta(\boldsymbol{p}-\boldsymbol{p}')\,\hbar^3 \int d\boldsymbol{\gamma}\ d\boldsymbol{\eta}\ \delta(\boldsymbol{r}-\boldsymbol{r}'+\tfrac{1}{2}\hbar\boldsymbol{\gamma})\, e^{i\boldsymbol{\gamma}\cdot(\boldsymbol{\eta}-\boldsymbol{p})}\ f_a(\boldsymbol{\eta},t) \\ &\qquad - f_a(x,t)\ f_b(x',t)\Big\} . \end{aligned} \tag{72.5}$$

In the classical limit this relation reduces to (26.10).

By the introduction of the exchange effects there appear new terms in (72,2), (72.4). We shall not write them down, as they will not be used below.

The averaging of Eqs. (71.8), (71.13) leads, as in the classical theory, to a hierarchy of equations. In the first moment approximation we obtain from (71.13), (71.11) the quantum self-consistent field equations for the distribution f_a and the function $U_a(\boldsymbol{r}, \boldsymbol{p}, t)$. Thus, for fermions we obtain

$$\left\{\frac{\partial}{\partial t} + \left(\boldsymbol{v} + \frac{\partial U_a}{\partial \boldsymbol{p}}\right)\cdot \frac{\partial}{\partial \boldsymbol{r}} - \frac{\partial U_a}{\partial \boldsymbol{r}} \cdot \frac{\partial}{\partial \boldsymbol{p}}\right\} f_a(x,t) = 0$$

$$U_a(\boldsymbol{r}, \boldsymbol{p}, t) = \sum_b N_b \frac{1}{(2\pi\hbar)^3} \int d\boldsymbol{r}'\, d\boldsymbol{p}'\, f_b(\boldsymbol{r}', \boldsymbol{p}', t)$$

$$\left[\Phi_{ab}(\boldsymbol{r}-\boldsymbol{r}') - \delta_{ab}\frac{1}{g_a} \nu_{aa}(\hbar^{-1}|\boldsymbol{p}-\boldsymbol{p}'|)\, \delta(\boldsymbol{r}-\boldsymbol{r}')\right]. \tag{72.6}$$

Equations of this kind appear in Landau's theory of Fermi liquids [19,20]. These equations have no fluctuation term, and thus do not describe dissipative processes.

73. THE POLARIZATION APPROXIMATION

In order to describe the processes in the polarization approximation we start from an equation analogous to the classical one, (27.7). As appears from (71.8), it has the form

$$\hat{L}_a\left\{\delta N_a(x,t) - \delta N_a^{\mathbf{source}}(x,t)\right\}$$
$$= \frac{i}{\hbar}\frac{N_a}{(2\pi)^3}\int d\boldsymbol{\gamma}\, d\boldsymbol{\eta}\left[\delta U_a(\boldsymbol{r}-\tfrac{1}{2}\hbar\boldsymbol{\gamma}) - \delta U_a(\boldsymbol{r}+\tfrac{1}{2}\hbar\boldsymbol{\gamma})\right] e^{i\boldsymbol{\gamma}\cdot(\boldsymbol{\eta}-\boldsymbol{p})} f_a(\boldsymbol{r},\boldsymbol{\eta},t) \tag{73.1}$$

where

$$\hat{L}_a A(\boldsymbol{r},\boldsymbol{p}) = \left(\frac{\partial}{\partial t} + \boldsymbol{v}\cdot\frac{\partial}{\partial \boldsymbol{r}}\right)A(\boldsymbol{r},\boldsymbol{p}) - \frac{i}{\hbar}\frac{1}{(2\pi)^3}\int d\boldsymbol{\gamma}\, d\boldsymbol{\eta}\, e^{i\boldsymbol{\gamma}\cdot(\boldsymbol{\eta}-\boldsymbol{p})}$$

$$\left[U_a(\boldsymbol{r}-\tfrac{1}{2}\hbar\boldsymbol{\gamma}) - U_a(\boldsymbol{r}+\tfrac{1}{2}\hbar\boldsymbol{\gamma})\right] A(\boldsymbol{r},\boldsymbol{\eta},t) \tag{73.2}$$

and U_a is the potential energy of the particles. For $\hbar\to 0$ (using (66.5)), (73.1), (73.2) reduce to (27.7).

To Eq. (73.1) must be added an equation relating the fluctuations δN_a, δU_a. In the absence of exchange effects, we get from (71.7)

$$\delta U_a(\boldsymbol{r},t) = \sum_b \frac{1}{(2\pi\hbar)^3}\int dx'\, \Phi_{ab}(\boldsymbol{r}-\boldsymbol{r}')\, \delta N_b(x',t). \tag{73.3}$$

The two-time correlation of the source fluctuations $\delta N_a^{\mathbf{source}}$ is defined by an equation similar to the classical one (27.8) :

$$\hat{L}_a^0 \left\langle \delta N_a\, \delta N_b\right\rangle^{\mathbf{source}}_{x,x',t,t'} = 0. \tag{73.4}$$

The initial condition for this equation follows from the expression of the one-time correlation $\langle \delta N_a \, \delta N_b \rangle_{x,x',t}$ for $g_{ab} = 0$. Thus, for a spatially homogeneous plasma we get from (72.5) :

$$\left\langle \delta N_a \; \delta N_b \right\rangle^{\text{source}}_{x,x',t'} \Big|_{t=t'} = N_a \delta_{ab}$$

$$\Big\{ \hbar^3 \, \delta(\boldsymbol{p}-\boldsymbol{p}') \int d\boldsymbol{\gamma} \; d\boldsymbol{\eta} \; \delta(\boldsymbol{r}-\boldsymbol{r}' + \tfrac{1}{2}\hbar\boldsymbol{\gamma}) \; e^{i\boldsymbol{\gamma}\cdot(\boldsymbol{\eta}-\boldsymbol{p})} f_a(\boldsymbol{\eta}, t) - f_a(x,t) \; f_b(x',t) \Big\} \,. \tag{73.5}$$

In the thermodynamic limit $N \to \infty$, $V \to \infty$, the second term involving $f_a f_b$ can be neglected. For $\hbar \to 0$ (73.5) reduces to (27.9).

74. THE QUANTUM COLLISION INTEGRAL IN THE POLARIZATION APPROXIMATION

Equations (73.1) – (73.4) can be taken as a basis for the calculation of the spectral densities of the fluctuations δN_a, δU_a. As the calculations are very similar to those of sections 34 – 37, we give immediately some results.

For an ideal plasma, i.e., without retardation and inhomogeneity effects, the spectral density of the field fluctuations is

$$\left(\delta \boldsymbol{E} \cdot \delta \boldsymbol{E}\right)_{\omega,\boldsymbol{k}} = \pi\hbar \frac{V}{(2\pi\hbar)^3} \sum_a \frac{(4\pi)^2 \; e_a^2 n_a}{k^2 |\varepsilon(\omega,\boldsymbol{k})|^2} \int d\boldsymbol{p}' \; d\boldsymbol{p}'' \; \delta(\hbar\boldsymbol{k} - \boldsymbol{p}' + \boldsymbol{p}'')$$
$$\delta\left(\hbar\omega - \frac{p'^2}{2m_a} + \frac{p''^2}{2m_a}\right)\Big[f_a(\boldsymbol{p}',t) + f_a(\boldsymbol{p}'',t)\Big] \,. \tag{74.1}$$

Here $\varepsilon(\omega,\boldsymbol{k})$ is the dielectric constant :

$$\varepsilon(\omega,\boldsymbol{k}) = 1 + \sum_a \frac{4\pi e_a^2 n_a}{k^2} \frac{V}{(2\pi\hbar)^3} \int d\boldsymbol{p}' \; d\boldsymbol{p}'' \; \delta(\hbar\boldsymbol{k} - \boldsymbol{p}' + \boldsymbol{p}'')$$
$$\frac{f_a(\boldsymbol{p}',t) - f_a(\boldsymbol{p}'',t)}{\hbar(\omega + i\Delta) - (p'^2/2m_a - p''^2/2m_a)} \tag{74.2}$$

For $\hbar \to 0$, Eqs (74.1), (74.2) reduce to the classical ones (35.6), (35.5).

In equilibrium, (74.1) becomes

$$\left(\delta \boldsymbol{E} \cdot \delta \boldsymbol{E}\right)_{\omega,\boldsymbol{k}} = \frac{8\pi}{\omega} \frac{\operatorname{Im} \varepsilon(\omega,\boldsymbol{k})}{|\varepsilon(\omega,\boldsymbol{k})|^2} \left(\frac{1}{2}\hbar\omega + \frac{\hbar\omega}{e^{\hbar\omega/k_B T} - 1}\right) \tag{74.3}$$

and coincides with the well-known expression of the equilibrium theory of electromagnetic fluctuations [21].

We rewrite the expression of the imaginary part of the dielectric constant which follows from (74.2) :

$$\operatorname{Im} \varepsilon(\omega,\boldsymbol{k}) = -\sum_a \frac{4\pi^2 \; e_a^2 n_a}{k^2} \frac{V}{(2\pi\hbar)^3} \int d\boldsymbol{p}' \; d\boldsymbol{p}'' \; \delta(\hbar\boldsymbol{k} - \boldsymbol{p}' + \boldsymbol{p}'')$$
$$\delta\left(\hbar\omega - \frac{p'^2}{2m_a} - \frac{p''^2}{2m_a}\right)\Big[f_a(\boldsymbol{p}',t) - f_a(\boldsymbol{p}'',t)\Big] \tag{74.4}$$

The quantum collision integral can be expressed as in the classical case (sect. 37) in terms of the functions $(\delta \boldsymbol{E}\cdot\delta\boldsymbol{E})_{\omega,\boldsymbol{k}}$ and $\varepsilon(\omega,\boldsymbol{k})$. We start from the equation for the one-particle distribution function (71.8). Using (73.2) we get

$$\hat{L}_a f_a = I_a(\boldsymbol{r},\boldsymbol{p},t) \tag{74.5}$$

where the collision integral is

$$I_a(\boldsymbol{r},\boldsymbol{p},t) = \frac{i}{\hbar}\frac{1}{(2\pi)^3}\int d\boldsymbol{\gamma}\, d\boldsymbol{\eta}\, e^{i\boldsymbol{\gamma}\cdot(\boldsymbol{\eta}-\boldsymbol{p})} \left\{\left\langle \delta U_a\, \delta N_a\right\rangle_{\boldsymbol{r}-\frac{1}{2}\hbar\boldsymbol{\gamma},\,\boldsymbol{r},\,\boldsymbol{\eta},\,t} - \left\langle \delta U_a\, \delta N_a\right\rangle_{\boldsymbol{r}+\frac{1}{2}\hbar\boldsymbol{\gamma},\,\boldsymbol{r},\,\boldsymbol{\eta},\,t}\right\} \tag{74.6}$$

Thus, the collision integral involves the correlations of δN_a, δU_a. The quantum collision integral can be expressed in terms of the spectral density $(\delta N_a\, \delta U_a)_{\omega,\boldsymbol{k},\boldsymbol{p}}$. The latter is provided by calculations similar to the classical ones of sections 36, 37. As a result we obtain for the collision integral of an ideal plasma :

$$I_a(\boldsymbol{p},t) = \frac{e_a^2}{(2\pi)^3\hbar}\int d\boldsymbol{p}'\, d\omega\, d\boldsymbol{k}\, \frac{1}{k^2}\,\delta(\hbar\boldsymbol{k}-\boldsymbol{p}+\boldsymbol{p}')\,\delta\left(\hbar\omega-\frac{p^2}{2m_a}+\frac{p'^2}{2m_a}\right) \left\{\left(\delta\boldsymbol{E}\cdot\delta\boldsymbol{E}\right)_{\omega,\boldsymbol{k}}\left[f_a(\boldsymbol{p}',t)-f_a(\boldsymbol{p},t)\right] - \frac{4\pi\hbar\, \mathrm{Im}\,\varepsilon(\omega,\boldsymbol{k})}{|\varepsilon(\omega,\boldsymbol{k})|^2}\left[f_a(\boldsymbol{p}',t)+f_a(\boldsymbol{p},t)\right]\right\} \tag{74.7}$$

In equilibrium the collision integral vanishes. Indeed, upon substituting the Maxwell distribution we find the identity

$$\delta\left[\hbar\omega-\frac{p^2}{2m_a}+\frac{p'^2}{2m_a}\right]\frac{1}{2}\,\frac{f_a(\boldsymbol{p}')+f_a(\boldsymbol{p})}{f_a(\boldsymbol{p}')-f_a(\boldsymbol{p})} = \delta\left(\hbar\omega-\frac{p^2}{2m_a}+\frac{p'^2}{2m_a}\right)\left(\frac{1}{2}+\frac{1}{e^{\hbar\omega/k_{\mathrm{B}}T}-1}\right) \tag{74.8}$$

and use Eq. (74.3). In the classical limit Eq. (74.7) reduces to the Balescu-Lenard collision integral in the form (37.2).

The quantum collision integral can be put into a form analogous to (37.1). We substitute the expressions (74.1), (74.4) into (74.7) :

$$I_a(\boldsymbol{p}_1,t) = \sum_b 4e_a^2 e_b^2 n_b \frac{V}{(2\pi\hbar)^3}\int d\boldsymbol{p}_1'\, d\boldsymbol{p}_2'\, d\boldsymbol{p}_2\, d\omega\, d\boldsymbol{k}\; \delta(\hbar\boldsymbol{k}-\boldsymbol{p}_1+\boldsymbol{p}_1')\,\delta\left(\hbar\omega-\frac{p_1^2}{2m_a}+\frac{p_1'^2}{2m_a}\right)\delta(\boldsymbol{p}_1+\boldsymbol{p}_2-\boldsymbol{p}_1'-\boldsymbol{p}_2') \;\delta\left(\frac{p_1^2}{2m_a}+\frac{p_2^2}{2m_b}-\frac{p_1'^2}{2m_a}-\frac{p_2'^2}{2m_b}\right)\frac{1}{k^4|\varepsilon(\omega,\boldsymbol{k})|^2} \left[f_a(\boldsymbol{p}_1',t)\,f_b(\boldsymbol{p}_2',t)-f_a(\boldsymbol{p}_1,t)\,f_b(\boldsymbol{p}_2,t)\right]. \tag{74.9}$$

We then perform the integrations over ω and $\boldsymbol{k}$ with the result :

$$I_a(\boldsymbol{p}_1,t) = \sum_b 4\pi e_a^2 e_b^2 n_b \frac{V}{(2\pi\hbar)^3} \int d\boldsymbol{p}_1' \, d\boldsymbol{p}_2' \, d\boldsymbol{p}_2 \; \delta(\boldsymbol{p}_1 + \boldsymbol{p}_2 - \boldsymbol{p}_1' - \boldsymbol{p}_2')$$

$$\delta\left(\frac{p_1^2}{2m_a} + \frac{p_2^2}{2m_b} - \frac{p_1'^2}{2m_a} - \frac{p_2'^2}{2m_b}\right)$$

$$\left\{(\boldsymbol{p}_1 - \boldsymbol{p}_1')^4 \left| \varepsilon\left[\hbar^{-1}\left(\frac{p_1^2}{2m_a} - \frac{p_1'^2}{2m_a}\right), \hbar^{-1}(\boldsymbol{p}_1 - \boldsymbol{p}_1')\right]\right|^2\right\}^{-1}$$

$$\left[f_a(\boldsymbol{p}_1',t)\, f_b(\boldsymbol{p}_2',t) - f_a(\boldsymbol{p}_1,t)\, f_b(\boldsymbol{p}_2,t)\right] . \tag{74.10}$$

The collision integral was obtained in this form by Silin [8] and by Balescu [7].

We now compare the collision operators in the polarization approximation, (74.10) and in the binary collision approximation (69.11). We therefore set in the latter $\nu_{ab}(k) = 4\pi e_a e_b / k^2$, which corresponds to Coulomb interactions. The difference between the two cases is the presence of the function $|\varepsilon|^{-2}$ in the former: this factor describes precisely the polarization. As the dielectric constant depends on the momenta, the polarization is called dynamical.

Still another form of (74.10) is useful. We introduce in the integrand the new variable $\boldsymbol{p}_2 - \boldsymbol{p}_2' = \hbar\boldsymbol{k}$, $(\boldsymbol{p}_2 + \boldsymbol{p}_2')/2 = \boldsymbol{p}$, hence

$$\boldsymbol{p}_2 = \boldsymbol{p} + \tfrac{1}{2}\hbar\boldsymbol{k} , \quad \boldsymbol{p}_2' = \boldsymbol{p} - \tfrac{1}{2}\hbar\boldsymbol{k} \; . \tag{74.11}$$

We then get

$$I_a(\boldsymbol{p}_1,t) = \sum_b \frac{4\pi e_a^2 e_b^2 n_b}{\hbar} \frac{V}{(2\pi\hbar)^3} \int d\boldsymbol{p}_1' \, d\boldsymbol{p} \, d\boldsymbol{k} \; \delta(\boldsymbol{p}_1 - \boldsymbol{p}_1' - \hbar\boldsymbol{k})$$

$$\delta\left(\frac{p_1^2}{2m_a} - \frac{p_1'^2}{2m_a} - \hbar\,\boldsymbol{k}\cdot\boldsymbol{v}\right) \frac{1}{k^4 |\varepsilon(\boldsymbol{k}\cdot\boldsymbol{v}, \boldsymbol{k})|^2}$$

$$\left[f_a(\boldsymbol{p}_1',t)\, f_b(\boldsymbol{p} - \tfrac{1}{2}\hbar\,\boldsymbol{k},t) - f_a(\boldsymbol{p},t)\; f_b(\boldsymbol{p} + \tfrac{1}{2}\hbar\boldsymbol{k},t)\right] . \tag{74.12}$$

We thus see that the effect of the polarization is the replacement

$$\nu_{ab}(k) \longrightarrow \frac{\nu_{ab}(k)}{|\varepsilon(\boldsymbol{k}\cdot\boldsymbol{v}, \boldsymbol{k})|^2} \; . \tag{74.13}$$

The polarization effects show up most clearly at large distances, of order r_D. At short distances the quantum collision integral obtained here describes the interactions in the weak-coupling approximation. We already pointed out that in constructing a kinetic theory for nonideal plasmas we need a collision integral taking account both the polarization effects and the strong interactions, but this is very difficult. Therefore, as in the classical theory (sect. 55.56), we limit ourselves to the consideration of the average dynamical polarization.

75. THE EFFECTIVE POTENTIAL. THE PSEUDO-POTENTIAL

The change (74.13) introduced by the polarization effects is analogous to (47.1). We recall that in the classical theory the effective potential, taking account of the average dynmaical polarization, is defined by Eqs (47,2), (47.5). The latter is more general and can be used to define the effective potential in the quantum theory as well. Thus

$$\tilde{\nu}_{ab}(\boldsymbol{k}) = \frac{e_a e_b}{\sum\limits_c e_c^2 n_c} \frac{1}{4\pi} \left(\delta E \cdot \delta \boldsymbol{E}\right)_{\boldsymbol{k},t} . \tag{75.1}$$

The expression of the spatial spectral density of the fluctuations follows from (74.1):

$$\left(\delta \boldsymbol{E} \cdot \delta \boldsymbol{E}\right)_{\boldsymbol{k},t} = \sum_a \frac{8\pi^2 e_a^2 n_a}{k^2} \frac{V}{(2\pi\hbar)^3} \int d\boldsymbol{p}' \, d\boldsymbol{p}'' \, \delta(\hbar\boldsymbol{k} - \boldsymbol{p}' + \boldsymbol{p}'')$$

$$\frac{f_a(\boldsymbol{p}',t) + f_a(\boldsymbol{p}'',t)}{\left|\varepsilon\left(\frac{p'^2}{2m_a\hbar} - \frac{p''^2}{2m_a\hbar}, \boldsymbol{k}\right)\right|^2} . \tag{75.2}$$

We go over to the new integration variables $\boldsymbol{p}' - \boldsymbol{p}'' = \hbar\boldsymbol{k}'$, $(1/2)(\boldsymbol{p}' + \boldsymbol{p}'') = \boldsymbol{p}$ and integrate over $\boldsymbol{k}'$:

$$\left(\delta \boldsymbol{E} \cdot \delta \boldsymbol{E}\right)_{\boldsymbol{k},t} = \sum_a \frac{8\pi^2 e_a^2 n_a}{k^2} \frac{V}{(2\pi\hbar)^3} \int d\boldsymbol{p}$$

$$\frac{f_a(\boldsymbol{p} + \frac{1}{2}\hbar\boldsymbol{k}, t) + f_a(\boldsymbol{p} - \frac{1}{2}\hbar\boldsymbol{k}, t)}{|\varepsilon(\boldsymbol{k}\cdot\boldsymbol{v}, \boldsymbol{k})|^2} . \tag{75.3}$$

From (75.1) and (75.3) follows the expression of the effective potential:

$$\tilde{\nu}_{ab}(\boldsymbol{k}) = \nu_{ab}(\boldsymbol{k}) \frac{1}{2} \frac{1}{\sum\limits_c e_c^2 n_c}$$

$$\cdot \sum_a e_a^2 n_a \frac{V}{(2\pi\hbar)^3} \int d\boldsymbol{p} \frac{f_a(\boldsymbol{p} + \frac{1}{2}\hbar\boldsymbol{k}, t) + f_a(\boldsymbol{p} - \frac{1}{2}\hbar\boldsymbol{k}, t)}{|\varepsilon(\boldsymbol{k}\cdot\boldsymbol{v}, \boldsymbol{k})|^2} . \tag{75.4}$$

For $\hbar = 0$ this relation reduces to the classical form (47.2).

In equilibrium, Eqs (75.2), (75.3) reduce to [22]:

$$\left(\delta \boldsymbol{E} \cdot \delta \boldsymbol{E}\right)_{\boldsymbol{k}} = 4\pi k_{\mathrm{B}} T \left\{\left(1 - \frac{1}{\varepsilon(0, \boldsymbol{k})}\right) + 2\sum_{l=1}^{\infty} \mathrm{Re}\left(1 - \frac{1}{\varepsilon(2\pi i \hbar^{-1} l k_{\mathrm{B}} T, \boldsymbol{k})}\right)\right\} . \tag{75.5}$$

The zero-frequency, equilibrium dielectric constant can be derived from (74.2):

$$\varepsilon(0,\boldsymbol{k}) = 1 + \sum_a \frac{4\pi e_a^2 n_a}{k^2 k_{\mathrm{B}} T} \, {}_1F_1\left(1, \frac{3}{2}, -\frac{\hbar^2 k^2}{8 m_a k_{\mathrm{B}} T}\right) . \tag{75.6}$$

where ${}_1F_1$ is the confluent hypergeometric function.

In the classical limit, ${}_1F_1(1, \frac{3}{2}, 0) = 1$ and the second term in (75.5) vanishes; thus

$$\left(\delta \boldsymbol{E} \cdot \delta \boldsymbol{E}\right)_{\boldsymbol{k}} = \frac{4\pi k_B T}{1 + r_D^2 k^2} . \qquad (75.7)$$

Substituting (75.7) into (75.1) we obtain (47.4), hence the Debye potential $\tilde{\Phi}_{ab}(\boldsymbol{r})$.

The quantum effects are important when $\lambda_a k \geqslant 1$ ($\lambda_a = \hbar/(2mk_B T)^{\frac{1}{2}}$). As $m_e \ll m_i$, the quantum effects are important only for the electrons.

The study of the quantum effective potential (75.4), to be found in ref. [23], shows that at small distances ($\lambda_e k \gg 1$) both terms of (75.5) give the same contribution $2\pi k_B T/(r_D^2 k^2)$; hence

$$\left(\delta \boldsymbol{E} \cdot \delta \boldsymbol{E}\right)_{\boldsymbol{k}} = \frac{4\pi k_B T}{r_D^2 k^2} , \quad \tilde{\nu}_{ab}(\boldsymbol{k}) = \frac{4\pi\, e_a e_b}{k^2} . \qquad (75.8)$$

Thus, at short distances ($\lambda_e k \gg 1$) the quantum effective potential reduces to the Coulomb potential. On the other hand, at large distances ($\lambda_e k \ll 1$) the quantum effects play no rôle, and $\tilde{\Phi}_{ab}(\boldsymbol{r})$ reduces to the Debye potential. Hence, the significant differences between classical and quantum effective potentials occur for intermediate distances, $\lambda_e k \sim 1$.

In the classical theory, we may express the two-body distribution in terms of the effective potential (47.5):

$$f_{ab}(\boldsymbol{r}) = \exp\left[-\tilde{\Phi}_{ab}(\boldsymbol{r})/k_B T\right] \qquad (75.9)$$

which reduces at short distances to the Boltzmann distribution, and at large distances coincides with the Debye distribution.

In the quantum case the situation is more complicated: the function $\rho_{ei}(\boldsymbol{r})$ cannot be expressed through the effective potential $\tilde{\Phi}_{ei}(\boldsymbol{r})$. This difficulty originates from the existence of bound states, which are not considered in (75.1),(75.2). The contribution of these states can be included by the use of the so-called pseudo-potential.

We assume that the degree of ionization of the plasma is so high that the neutral atoms can be described as an ideal gas. The nonideality is due only to the interactions of the charged particles. The bound states can then be excluded from the consideration, and we may introduce a pseudo-potential through a formula of classical form:

$$\rho_{ei}(\boldsymbol{r}) = \exp\left(-\tilde{u}_{ei}/k_B T\right) . \qquad (75.10)$$

In ref. [23] an expression for the pseudo-potential was derived; for large distances ($r \sim r_D$) it tends to the effective Debye potential, for intermediate distances ($a_0 \ll r \ll r_D$, where a_0 is the Bohr radius) it is close to the Coulomb potential, and for $r = 0$ it tends to a finite value. The simplest expression for the pseudo-potential is

$$\tilde{u}_{ei}(r) = -e^2\left(\frac{d}{a_0+cr}+\frac{fr}{a_0^2+r^2}\right)-\frac{e^2}{r}\left(e^{-r/r_D}-1\right) . \tag{75.11}$$

The three constants d, f, c, are defined by

$$d = \frac{a_0}{e^2}k_B T\left(\frac{\xi_{ei}^2}{16}+\ln\frac{\pi^{\frac{1}{2}}}{8}\xi_{ei}^3\right)$$

$$dc-f = \frac{a_0}{e^2}k_B T\frac{a_0}{\lambda_{ei}}\xi_{ei}$$

$$dc+f = 1 . \tag{75.12}$$

Here $\xi_{ab} = -e_a e_b/k_B T\lambda_{ab}$ is the Born parameter,

$$\lambda_{ab} = \mu_{ab}^{-\frac{1}{2}}\hbar\left(2k_B T_a/m_a + 2k_B T_b/m_b\right)^{-\frac{1}{2}} ,$$

and a_0 is the Bohr radius.

From (75.11) follows that the pseudo-potential has the following finite value for $r=0$:

$$\tilde{u}_{ei}(r=0) = -k_B T\left[\frac{\xi_{ei}^2}{16}+\ln\frac{\pi^{\frac{1}{2}}}{8}\xi_{ei}^3-\frac{\xi_{ei}\lambda_{ei}}{r_D}\right] . \tag{75.13}$$

For $r \gg a_0$ we find

$$\tilde{u}_{ei}(r) = \tilde{\Phi}_{ei}(r) = -\frac{e}{r}e^{-r/r_D} . \tag{75.14}$$

Note that in the absence of the polarization $(r_D=\infty)$, Eq. (75.11) behaves at large distances as $1/r^2$. It is well known that the Slater sum $S_{ei}(r)$, defining the function ρ_{ei} in that case, behaves for Coulomb systems as $1/r^4$. Eq. (75.11) can be improved in such a way that the terms in $1/r^2$, $1/r^3$ disappear. Then (75.11) behaves at large distances as $1/r^4$, in the absence of polarization [23].

The pseudo-potential (75.11) must be viewed only as a rather simple model expression, useful for a certain type of problem. The more complete theory of the pseudo-potential for a plasma is quite complicated, and is not very well developed at present.

76. THE QUANTUM BOLTZMANN EQUATION FOR A NONIDEAL PLASMA

The quantum kinetic equation for an ideal plasma in the polarization approximation was derived in section 74. As in the classical case (sects 53, 54) we may derive a generalized kinetic equation for nonideal plasmas. However, we know from section 56 that the contribution of the short-range interactions in the polarization approximation is incorrect. This difficulty is especially serious in the calculation of the non-dissipative properties of the plasma.

It is therefore necessary to derive a kinetic equation for nonideal plasmas taking into account both the polarization effects and the short-range strong interactions. As in the classical case we shall only consider this problem in the

approximation of the averaged dynamical polarization. This amounts to introducing the effects of the polarization through the use of the effective potential, or of the pseudo-potential.

In this approximation we may use for the two-particle density matrix ρ_{ab} the expression (66.10) obtained in the binary collision approximation, after substituting

$$\Phi_{ab} \longrightarrow u_{ab} \tag{76.1}$$

The potential u_{ab} is defined as follows

$$u_{ab} = \begin{cases} \tilde{\Phi}_{ab} & \text{for} \quad a = b \\ \tilde{u}_{ab} & \text{for} \quad a \neq b \end{cases} \tag{76.2}$$

Thus, the potential u_{ab} is the effective potential for particles of the same sign, and is the pseudo-potential for particles of different sign. If we limit ourselves to the equilibrium approximation, we may take for $\tilde{\Phi}_{ab}$ the Debye potential. and for $\tilde{u}_{ab}$ the expression (75.11).

We denote the density matrix obeying Eq. (66.10) with the potential (76.2) by $\tilde{\rho}_{ab}$. We express this density matrix in terms of the eigenfunctions $\tilde{\Psi}_{\boldsymbol{p}_1\boldsymbol{p}_2}(\boldsymbol{r}_1, \boldsymbol{r}_2)$ defined by (66.11) and (76.1). The solution for $\tilde{\rho}_{ab}$ can again be expressed in the form (66.19). The substitution of this result into $(66.8)_2$ provides the collision integral which can be represented in the form (66.20). For I_{a1} we find instead of (66.22) :

$$\begin{aligned} I_{a1}(\boldsymbol{p}_1, t) = \frac{i}{\hbar} \frac{V^2}{(2\pi)^6 \hbar^3} \sum_b N_b \int d\boldsymbol{\Upsilon}\, d\boldsymbol{p}_1'\, d\boldsymbol{p}_2\, d\boldsymbol{r}_2\, e^{-i\boldsymbol{\Upsilon}\cdot\boldsymbol{p}_1} \\ \left[\Phi_{ab}(\boldsymbol{r}_1 - \boldsymbol{r}_2 - \tfrac{1}{2}\hbar\boldsymbol{\Upsilon}) - \Phi_{ab}(\boldsymbol{r}_1 - \boldsymbol{r}_2 + \tfrac{1}{2}\hbar\boldsymbol{\Upsilon})\right] \\ \tilde{\Psi}_{\boldsymbol{p}_1'\boldsymbol{p}_2}(\boldsymbol{r}_1 + \tfrac{1}{2}\hbar\boldsymbol{\Upsilon}, \boldsymbol{r}_2)\, \tilde{\Psi}^{*}_{\boldsymbol{p}_1'\boldsymbol{p}_2}(\boldsymbol{r}_1 - \tfrac{1}{2}\hbar\boldsymbol{\Upsilon}, \boldsymbol{r}_2)\, f_a(\boldsymbol{p}_1', t)\, f_b(\boldsymbol{p}_2, t) \end{aligned} \tag{76.3}$$

where $\tilde{\Psi}$ is an eigenfunction obeying the equation

$$\left[-\frac{\hbar^2}{2m_a}\nabla^2_{\boldsymbol{r}_1} - \frac{\hbar^2}{2m_b}\nabla^2_{\boldsymbol{r}_2} + u_{ab}(\boldsymbol{r}_1 - \boldsymbol{r}_2)\right]\tilde{\Psi}_{\boldsymbol{p}_1\boldsymbol{p}_2}(\boldsymbol{r}_1, \boldsymbol{r}_2) = E_{\boldsymbol{p}_1\boldsymbol{p}_2}\tilde{\Psi}_{\boldsymbol{p}_1\boldsymbol{p}_2}(\boldsymbol{r}_1, \boldsymbol{r}_2)\,. \tag{76.4}$$

Equation (76.3) cannot be brought into the form of a Boltzmann equation with a quantum cross-section, because it contains collective interactions through the potential u_{ab}. The solution of Eq. (76.4) is as yet unknown. In ref. [24] some numerical data about the eigenvalues of this equation were obtained for a Debye potential.

77. THERMODYNAMIC FUNCTIONS OF A NONIDEAL PLASMA

The purpose of this section is to examine to what degree the expressions for the thermodynamic functions obtained from the models considered above compare

to the known results of equilibrium quantum statistical mechanics. Till now, the concrete calculations were performed only with an effective Debye potential $\tilde{\Phi}_{ab}$, not with the potential u_{ab}.

From (67.10) we obtain an expression for the correlation part of the internal energy $U_{\mathbf{cor}}$ in terms of the potential energy. The corresponding expression for the free energy is

$$F_{\mathbf{cor}} = \int_0^1 d\lambda \, \frac{1}{\lambda} \, U_{\mathbf{pot}}(\beta,\lambda) \, , \quad U_{\mathbf{cor}} = \frac{\partial}{\partial \beta} \, \beta F_{\mathbf{cor}} \, . \tag{77.1}$$

From (67.1) we obtain

$$U_{\mathbf{pot}} = \tfrac{1}{2} \sum_a \sum_b n_a n_b \int d(\boldsymbol{r}_1 - \boldsymbol{r}_2) \, \Phi_{ab}(\boldsymbol{r}_1 - \boldsymbol{r}_2) \, \tilde{\rho}^1_{ab}(\boldsymbol{r}_1, \boldsymbol{r}_1, \boldsymbol{r}_2, \boldsymbol{r}_2) \, . \tag{77.2}$$

The function $\tilde{\rho}^1_{ab}$ is given by (67.2), with $\Psi \to \tilde{\Psi}$. The exact form of $\tilde{\Psi}_{\boldsymbol{p}_1 \boldsymbol{p}_2}$ is unknown, even for the Debye potential. We therefore express the thermodynamic functions as power series in the Born parameter ξ_{ab}.

In the Born approximation, and for the local-equilibrium state, the Fourier transform of $\tilde{\rho}^1_{ab}$ is given by

$$\tilde{\rho}^1_{ab}(\boldsymbol{k}) = (2\pi)^3 \, \delta(\boldsymbol{k}) - \frac{m_a + m_b}{m_a k_{\mathbf{B}} T_b + m_b k_{\mathbf{B}} T_a} \, u_{ab}(\boldsymbol{k}) \, {}_1F_1\left(1 \, , \frac{3}{2} , \, -\frac{1}{4} \, \lambda^2_{ab} k^2\right) \tag{77.3}$$

From (77.2), (77.3) we find

$$U_{\mathbf{pot}} = - \, 2\pi \sum_a \sum_b n_a n_b e_a^2 e_b^2 \, \frac{m_a + m_b}{m_a k_{\mathbf{B}} T_b + m_b k_{\mathbf{B}} T_a} \, r_{\mathbf{D}} \, G(\lambda_{ab}/r_{\mathbf{D}}) \, , \tag{77.4}$$

where

$$G(\eta) = \frac{1}{\pi e_a e_b r^2_{\mathbf{D}}} \int_0^\infty d\rho \, \rho^2 \int_0^\infty dk' \, u_{ab}(k') \, e^{-\rho^2} \left[\frac{2\rho}{\eta k'} \sin(\eta k' \rho) - \cos(\eta k' \rho)\right] \tag{77.5}$$

with $k' = r_{\mathbf{D}} k$. The expressions (77.4), (77.5) are valid for any potential u_{ab}. For the case of the effective Debye potential

$$G(\eta) = \frac{\pi^{\frac{1}{2}}}{\eta} \left\{1 - \exp(\eta^2/4) \, [1 - \Phi(\eta/2)]\right\} \, . \tag{77.6}$$

For $\eta \ll 1$ we find

$$G\left(\frac{\lambda_{ab}}{r_{\mathbf{D}}}\right) = 1 - \frac{1}{4} \, \pi^{\frac{1}{2}} \, \frac{\lambda_{ab}}{r_{\mathbf{D}}} + 0\left(\frac{\lambda^2_{ab}}{r^2_{\mathbf{D}}}\right) \, . \tag{77.7}$$

For an isothermal plasma we find from (77.7), (77.4) an expression identical with the result of the equilibrium theory [25, 5, 44, 39].

Thus, to first order in $\lambda_{ab}/r_{\mathbf{D}}$, the use of the Debye effective potential leads to the correct equilibrium expressions. Note also that (77.7) is practically independent of the choice of the effective potential.

For an isothermal plasma in the local-equilibrium approximation and for $(\lambda_{ab}/r_D) \ll 1$ the calculation of the thermodynamic functions can be performed to arbitrary order in the Born parameter. For the internal energy density we obtain

$$U = \frac{3}{2} k_B T \sum_a n_a - \frac{k_B T}{8\pi r_D^3}$$

$$- \pi k_B T \sum_a \sum_b n_a n_b \lambda_{ab}^3 \left\{ -\frac{5}{8} \pi^{\frac{1}{2}} \xi_{ab}^2 - \left(a + \ln \frac{\lambda_{ab}}{r_D} \right) \xi_{ab}^3 \right.$$

$$\left. + \pi^{\frac{1}{2}} \sum_{m=4}^{\infty} (m+3) \frac{\zeta(m-2)}{\Gamma(\frac{1}{2}m)+1} \left(\frac{1}{2} \xi_{ab} \right)^m + 0 \left(\frac{\lambda_{ab}}{r_D} \right) \right\} , \tag{77.8}$$

where

$$a = \frac{1}{2} C + \ln 2 - \frac{1}{3}$$

and ζ is the Riemann zeta-function, Γ the gamma function, and C the Euler constant.

This result agrees with the result of equilibrium theory. The only difference is in the value of the constant a, but this difference is not important, as in a plasma with $e_e = - e_i$ the term in ξ_{ab}^3 vanishes.

Equation (77.8) takes account implicitly of the bound states with a lifetime much shorter than the mean free time. In order to exhibit this contribution more explicitly, we transform (77.8) further, for the case of a hydrogen plasma. We make use of the following formula [39]:

$$\lambda_{ie}^3 \sum_{m=2}^{\infty} (2m+3) \frac{\zeta(2m-2)}{\Gamma(m)+1} \left(\frac{1}{2} \xi_{ie} \right)^{2m}$$

$$= \frac{\partial}{\partial \beta} \left\{ \lambda_{ie}^3 \sum_{s=2}^{\infty} \frac{1}{s!} \left(\frac{1}{4} \xi_{ie}^2 \right)^s \sum_{n=1}^{\infty} \frac{1}{n^{2s-1}} \right\}$$

$$= \frac{\partial}{\partial \beta} \left\{ \lambda_{ie}^3 \sum_{n=1}^{\infty} n^2 \left[e^{-\beta E_n} - 1 + \beta E_n \right] \right\} \tag{77.9}$$

where we used the fact that $\xi_{ie}^2/4n^2 = - \beta E_n$, E_n being the levels of the hydrogen atom. This equation gives the total contribution to (77.8) of the even powers of ξ_{ei}, beginning with ξ_{ei}^4: this is the contribution of the bound states. We see that it is defined as the derivative of the partition function of the hydrogen atom, cut off by the Planck-Larkin method [28, 29].

The remaining contribution of the ξ_{ei}-series will be denoted by $K^*(\beta)$. Thus, the expression of the internal energy of the plasma can be written in the form:

$$U = \frac{3}{2}\, k_B T \sum_a n_a - \frac{k_B T}{8\pi r_D^3}$$

$$- n_e^2 \frac{\partial}{\partial\beta} \left\{ 8\,\pi^{\frac{3}{2}} \lambda_{ie}^3 \sum_{n=1}^{\infty} n^2 \left[e^{-\beta E_n} - 1 + \beta E_n \right] + K^*(\beta) \right\}. \tag{77.10}$$

The contribution K^* depends weakly on the temperature. It can be approximated by the following interpolation formula :

$$K^*(\beta) \approx -4\pi^{\frac{3}{2}} \lambda_{ie}^3 |\beta E_1|$$

where E_1 is the ground-state energy of the hydrogen atom.

We now come back to the two-particle distribution function

$$\tilde{\rho}_{ab}^1(\boldsymbol{r}_1, \boldsymbol{r}_1, \boldsymbol{r}_2, \boldsymbol{r}_2).$$

This function is finite for all distances $r = |\boldsymbol{r}_1 - \boldsymbol{r}_2|$. Its decay at large distances is determined by the polarization, whereas its finiteness at short distances is due to the quantum effects. For instance, in the local, non-isothermal equilibrium state, we find from (77.3) by using the Debye potential :

$$\tilde{\rho}_{ab}^1(r=0) = 1 - \frac{\pi^{\frac{1}{2}} (m_a + m_b)\, e_a e_b}{(m_a k_B T_b + m_b k_B T_a)\, \lambda_{ab}} \exp\left(\frac{\lambda_{ab}^2}{4 r_D^2}\right) \left[1 - \Phi\left(\frac{\lambda_{ab}}{2 r_D}\right)\right]. \tag{77.11}$$

For an isothermal plasma, for $\lambda_{ab} r_D \ll 1$, this reduces to

$$\tilde{\rho}_{ab}^1(r=0) = 1 - \pi^{\frac{1}{2}} \frac{e_a e_b}{\lambda_{ab} k_B T} \tag{77.12}$$

which is identical to the result of the works of Kelbg [30] and of Trubnikov and Ebeling [25, 44].

78. EXCHANGE EFFECTS

We now discuss some results of the quantum kinetic theory of nonideal plasmas, including exchange effects. The quantum kinetic Boltzmann equation for a nonideal plasma, including exchange effects was considered in ref. [6]. This equation is obtained from the equations of section 70 by the method described in section 76. We consider some of these results in order to illustrate the exchange effects.

If the exchange terms are included, Eq. (77.3) takes the form :

$$\begin{aligned}
\tilde{\rho}_{ab}^1(\boldsymbol{k}) &\equiv \sum_{\sigma_1} \sum_{\sigma_2} \tilde{\rho}_{ab}^1(\boldsymbol{\sigma}_1, \boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2, \boldsymbol{\sigma}_2, \boldsymbol{k}) \\
&= (2\pi)^3 \delta(\boldsymbol{k}) - \frac{1}{k_B T} u_{ab}(\boldsymbol{k})\, {}_1F_1\left(1, \tfrac{3}{2}, \tfrac{1}{2}\lambda_{ab}^2 k^2\right) \\
&- \delta_{ab} \frac{\pi^{\frac{3}{2}}}{g_a} \lambda_{ab}^{\frac{3}{2}} \exp\left(-\tfrac{1}{4}\lambda_{ab}^2 k^2\right) \\
&+ \delta_{ab} \frac{\lambda_{ab}}{g_a \pi^{\frac{3}{2}} k_B T} \exp\left(-\tfrac{1}{4}\lambda_{ab}^2 k^2\right) \int_0^\infty dt\, \exp\left(-\lambda_{ab}^2 t^2\right) u_{ab}(2t)\, \frac{sh\, \lambda_{ab}^2 k t}{k t}
\end{aligned} \tag{78.1}$$

This equation is valid for any function u_{ab}. The exchange terms in this equation are those involving the factor δ_{ab}. For a Coulomb potential the last term in (78.1) is

$$\frac{\delta_{ab}}{g_a}\,\frac{2\pi e_a e_b \lambda_{ab}^2}{k_B T}\,\exp\left(-\tfrac{1}{4}\lambda_{ab}^2 k^2\right)\sum_{\nu=0}^{\infty}\frac{(\lambda_{ab}k)^{2\nu}}{4^\nu\,\nu!\,(2\nu+1)^2}\,.$$

The distribution function $\tilde{\rho}_{ab}^1$ for a Coulomb system is (Kelbg [30]):

$$\begin{aligned}\tilde{\rho}_{ab}^1(r) &= 1 - \frac{\delta_{ab}}{g_a}\,e^{-r^2/\lambda_{ab}^2}\\ &\quad - \frac{e_a e_b}{k_B T}\left\{\frac{1}{r}\left(1 - e^{-r^2/\lambda_{ab}^2}\right) + \frac{\pi^{\frac{1}{2}}}{\lambda_{ab}}\left[1 - \Phi\left(\frac{r}{\lambda_{ab}}\right)\right]\right\}\\ &\quad + \frac{\delta_{ab}}{g_a}\,\frac{e_a e_b}{k_B T}\,\frac{1}{r}\,e^{-r^2/\lambda_{ab}^2}\int_0^1 d\alpha\,\frac{1}{\alpha}\,\Phi\left(\frac{\alpha r}{\lambda_{ab}(1-\alpha)^{\frac{1}{2}}}\right)\end{aligned} \tag{78.2}$$

For $r=0$ and $g_a = 2$ we find

$$\tilde{\rho}_{ab}^1(\boldsymbol{r}=0) = \left(1 - \frac{\pi^{\frac{1}{2}}\,e_a e_b}{k_B T\,\lambda_{ab}}\right)\left(1 - \frac{\delta_{ab}}{2}\right) \tag{78.3}$$

Thus, for $a=b$, the result is divided by two, as a result of the exchange effects.

For the Debye potential we find from (77.3) or (78.1) without exchange effects the result :

$$\begin{aligned}\tilde{\rho}_{ab}^1(r) &= 1 - \frac{e_a e_b\,\pi^{\frac{1}{2}}\,r_D}{2\,k_B T\,\lambda_{ab}}\,\frac{1}{r}\\ &\quad\cdot\left\{\exp\left(-\frac{r}{r_D} + \frac{1}{4}\frac{\lambda_{ab}^2}{r_D^2}\right)\left[\Phi\left(\frac{r}{\lambda_{ab}} - \frac{\lambda_{ab}}{2r_D}\right) + 2\Phi\left(\frac{\lambda_{ab}}{2r_D}\right) - 1\right]\right.\\ &\quad\left. + \exp\left(\frac{r}{r_D} + \frac{1}{4}\frac{\lambda_{ab}^2}{r_D^2}\right)\left[1 - \Phi\left(\frac{r}{\lambda_{ab}} + \frac{\lambda_{ab}}{2r_D}\right)\right]\right\}\end{aligned} \tag{78.4}$$

The contribution of the exchange terms is complicated: we only write the result for $r=0$ $(g_a = 2)$:

$$\tilde{\rho}_{ab}^1(r=0) = \left\{1 - \pi^{\frac{1}{2}}\,\frac{e_a e_b}{k_B T\,\lambda_{ab}}\,\exp\left(\frac{1}{4}\frac{\lambda_{ab}^2}{r_D^2}\right)\left[1 - \Phi\left(\frac{\lambda_{ab}}{2r_D}\right)\right]\right\}\left(1 - \frac{\delta_{ab}}{2}\right) \tag{78.5}$$

We see that here too the result is divided by two for $a=b$ as a result of the exchange effects.

We now consider the thermodynamic functions, including the exchange effects. We know that

$$\begin{aligned}U_{\mathbf{pot}} &= \tfrac{1}{2}\sum_a\sum_b n_a n_b\int d\boldsymbol{r}\;\tilde{\rho}_{ab}^1(\boldsymbol{r})\;\Phi_{ab}(\boldsymbol{r})\\ &= \tfrac{1}{2}\sum_a\sum_b n_a n_b\,\frac{1}{(2\pi)^3}\int d\boldsymbol{k}\;\tilde{\rho}_{ab}^1(\boldsymbol{k})\;\nu_{ab}(\boldsymbol{k})\,,\end{aligned} \tag{78.6}$$

as a result of Eqs. (67.10), (77.1). Here $\Phi_{ab}(r)$ is the Coulomb potential.

The part of the potential energy due to exchange effects is given by (78.1) with the Debye potential :

$$U_{\mathbf{pot}}^{\mathbf{exch}} = \frac{1}{2}\sum_a\sum_b n_a n_b \frac{\delta_{ab}}{g_a}\Bigg\{-2\pi\, e_a e_b \lambda_{ab}^2 + \frac{8\lambda_{ab}^2 e_a^2 e_b^2}{k_{\mathbf{B}}T}\int_0^1 du\int_0^1 dv\left[\frac{\pi^{\frac{1}{2}}}{2\lambda_{ab}(1-u^2v^2)^{\frac{1}{2}}} - \frac{\pi}{4r_{\mathbf{D}}}\exp\left[\frac{\lambda_{ab}^2}{4r_{\mathbf{D}}^2}\left(1-u^2v^2\right)\right]\cdot\left\{1-\Phi\left(\frac{\lambda_{ab}(1-u^2v^2)^{\frac{1}{2}}}{2r_{\mathbf{D}}}\right)\right\}\right]\Bigg\} . \tag{78.7}$$

The first bracketed term is the Hartree-Fock contribution. The first integral in (78.7) can be calculated exactly, the second one is expanded in powers of $\lambda_{ab}/r_{\mathbf{D}}$:

$$U_{\mathbf{pot}}^{\mathbf{exch}} = \frac{1}{2}\sum_a\sum_b n_a n_b \frac{\delta_{ab}}{g_a}\left\{-2\pi\, e_a e_b \lambda_{ab}^2 + \frac{2\pi\lambda_{ab} e_a^2 e_b^2}{k_{\mathbf{B}}T}\left[\pi^{\frac{1}{2}}\ln 2 - \frac{\lambda_{ab}}{r_{\mathbf{D}}}\right]\right\} \tag{78.8}$$

For $r_{\mathbf{D}}\to\infty$ (i.e., the Coulomb plasma) this result agrees with the result of equilibrium theory.

As in section 77 the thermodynamic functions, including the exchange effects, can be evaluated to arbitrary order in the Born parameter $\xi_{ab} = -e_a e_b/k_{\mathbf{B}}T\lambda_{ab}$. Thus

$$F_{\text{cor}} = -\frac{k_{\mathbf{B}}T}{12\pi r_{\mathbf{D}}^3} - 2\pi k_{\mathbf{B}}T\sum_a\sum_b n_a n_b \lambda_{ab}^3\left\{-\frac{\delta_{ab}}{g_a}E(\xi_{ab}) - \frac{1}{6}\xi_{ab}^3\ln\frac{\lambda_{ab}}{r_{\mathbf{D}}} + Q(\xi_{ab})\right\} . \tag{78.9}$$

The first term corresponds to the Debye-Hückel approximation. The term involving $E(\xi_{ab})$ describes exchange effects. This function is given by

$$E(\xi_{ab}) = \frac{1}{2}\xi_{ab} + \frac{\ln 2}{4}\pi^{\frac{1}{2}}\xi_{ab}^2 + \pi^{\frac{1}{2}}\sum_{m=3}^{\infty}\left(1-2^{2-m}\right)\frac{\zeta(m-1)}{\Gamma\left(\frac{m}{2}+1\right)}\left(\frac{\xi_{ab}}{2}\right)^m + 0\left(\xi_{ab}^2\,\frac{\lambda_{ab}}{r_{\mathbf{D}}}\right) \tag{78.10}$$

The last two terms in (78.9) describe the ordinary quantum effects. the function $Q(\xi_{ab})$ is defined as follows :

$$Q(\xi_{ab}) = -\frac{1}{8}\,\pi^{\frac{1}{2}}\,\xi_{ab}^{2} - \frac{a}{6}\,\xi_{ab}^{3} + \pi^{\frac{1}{2}} \sum_{m=4}^{\infty} \frac{\zeta(m-2)}{\Gamma(\frac{1}{2}m+1)} \left(\frac{\xi_{ab}}{2}\right)^{m} + 0\left(\xi_{ab}^{2}\,\frac{\lambda_{ab}}{r_{\mathrm{D}}}\right) \tag{78.11}$$

where $a = \frac{1}{2}\,C + \ln 3 - \frac{1}{2}$, C being the Euler constant.

We briefly presented some problems of the quantum theory of nonideal gases and plasmas. Because of lack of space, it is not possible to discuss here the quantum theory of kinetic fluctuations. We only note that this theory can be constructed by analogy with the corresponding classical theory.

CHAPTER 14

Kinetic Equations for Partially Ionized Plasmas and for Chemically Reacting Gases

79. PARTIALLY IONIZED PLASMAS. INITIAL MODEL

In recent times, the kinetic theory of molecular gases was thoroughly studied (see for instance the work of Dahler and Hoffmann [31] and of Waldmann [32]). The complication of the theory in that case is due to the necessity of including the internal degrees of freedom of the molecules in the description.

The kinetic theory of chemically reacting gases and of partially ionized plasmas is even more complex. Very few works have been devoted to the kinetic equations taking into account chemical transformations.

The work of Peletminsky [33] and of Kolesnichenko [34] represent an attempt at constructing such kinetic equations. They use an expansion in the density parameter, but modify the condition of weakening of the initial correlations. This method is not convenient when the concentration of bound particles increases.

Yakub [35] and Kolesnichenko [36] used a generalization of the Bogolyubov method for the derivation of kinetic equations for chemically reacting gases. The starting point is a Liouville equation for the distribution of N simple particles which can be transformed into bound complexes, the molecules. The difference with the Liouville equation (1.2) lies in the fact that the interaction potential cannot be considered to be additive. As a result, the right-hand sides of the equations of the hierarchy are infinite series in the density.

In order to derive kinetic equations including chemical reactions the density expansions must be cut off in such a way that, in each approximation, all possible collisions of any given group of particles be accounted for. Such a method was used in Kolesnichenko's work [36] †.

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† **A different, more fundamental approach to the problem of composite particles appears in refs. [50, 51, 52] . (Transl.)**

In refs. [22 ,27] a different method was used for the derivation of kinetic equations for partially ionized plasmas with Coulomb [22] and with electromagnetic interactions [37]. We essentially follow this method here.

We consider a three-component plasma, consisting of electrons, ions and hydrogen atoms. The electrons and the ions will be denoted by subscripts e and i respectively, whereas the atoms are denoted by ei. The charged particles interact through Coulomb forces.

In a partially ionized gas, there are two limiting situations: the state with zero degree of ionization, i.e., a neutral gas, and the state of a fully ionized plasma.

As a starting point we take the first state, in which all particles are bound as atoms. This state is characterized by a density matrix for the atoms, $\hat{\rho}_{ei}(\boldsymbol{r}'_e, \boldsymbol{r}'_i, \boldsymbol{r}''_e, \boldsymbol{r}''_i, t)$ (see (71.3)).

Instead of $\boldsymbol{r}_e, \boldsymbol{r}_i$, it is more convenient to use the variables :

$$\boldsymbol{R} = \frac{m_e \boldsymbol{r}_e + m_i \boldsymbol{r}_i}{m_e + m_i} \qquad \boldsymbol{r}_e = \boldsymbol{R} + \frac{m_i \boldsymbol{r}}{m_e + m_i}$$

$$\boldsymbol{r} = \boldsymbol{r}_e - \boldsymbol{r}_i \qquad \boldsymbol{r}_e = \boldsymbol{R} - \frac{m_e \boldsymbol{r}}{m_e + m_i} \tag{79.1}$$

and the corresponding density matrix $\hat{\rho}_{ei}(\boldsymbol{R}', \boldsymbol{R}'', \boldsymbol{r}', \boldsymbol{r}'', t)$. The latter obeys the equation

$$i\hbar \frac{\partial \hat{\rho}_{ei}}{\partial t} = -\frac{\hbar^2}{2M}\left(\nabla^2_{\boldsymbol{R}'} - \nabla^2_{\boldsymbol{R}''}\right)\hat{\rho}_{ei} - \frac{\hbar^2}{2\mu}\left(\nabla^2_{\boldsymbol{r}'} - \nabla^2_{\boldsymbol{r}''}\right)\hat{\rho}_{ei}$$

$$+ \left[\Phi_{ei}(\boldsymbol{r}') + \hat{U}_{ei}(\boldsymbol{R}', \boldsymbol{r}') - \Phi_{ei}(\boldsymbol{r}'') - \hat{U}_{ei}(\boldsymbol{R}'', \boldsymbol{r}'')\right]\hat{\rho}_{ei}\,. \tag{79.2}$$

Here Φ_{ei} is the Coulomb potential.

$$M = m_e + m_i\,, \qquad \mu_{ei} \equiv \mu = \frac{m_e m_i}{m_e + m_i}\,. \tag{79.2}$$

The function

$$\hat{U}_{ei}(\boldsymbol{R}, \boldsymbol{r}) = e_e \hat{U}(\boldsymbol{r}_e) + e_i \hat{U}(\boldsymbol{r}_i)$$

is the pptential energy of the particles in the pair, due to the field of the surrounding particles. The operator $\hat{U}(\boldsymbol{q})$ obeys the Poisson equation :

$$\nabla^2 \hat{U}(\boldsymbol{q}, t) = -4\pi \frac{1}{V^2}\int d\boldsymbol{R}\, d\boldsymbol{r}\left\{e_e\,\delta\left(\boldsymbol{q} - \boldsymbol{R} - \frac{m_i}{m_e + m_i}\boldsymbol{r}\right)\right.$$

$$\left. + e_i\,\delta\left(\boldsymbol{q} - \boldsymbol{R} + \frac{m_e}{m_e + m_i}\boldsymbol{r}\right)\right\}\hat{\rho}_{ei}(\boldsymbol{R}, \boldsymbol{R}, \boldsymbol{r}, \boldsymbol{r}, t)\,. \tag{79.4}$$

Equations (79.2) and (79.4) form a closed set for the operators $\hat{\rho}_{ei}$ *and* $\hat{U}$. We must add the normalization condition

$$\frac{1}{V^2}\int d\boldsymbol{r}\, d\boldsymbol{R}\, \hat{\rho}_{ei}(\boldsymbol{R}, \boldsymbol{R}, \boldsymbol{r}, \boldsymbol{r}) = N \tag{79.5}$$

where N is the total number of atoms for zero ionization degrees.

In the opposite limiting case, when all the charged particles are free, the starting point is the set (71.6), (71.7) for the operators $\hat{N}_a(x,t)$, $\hat{U}(\boldsymbol{r},t)$ (or for $\hat{\rho}_a$, $\hat{U}$).

We represent $\hat{\rho}_{ei}$ in terms of the eigenfunctions of the Hamiltonian of pairs, i.e., in terms of the eigenfunctions defined by:

$$\left[-\frac{\hbar^2}{2M}\nabla^2_{\boldsymbol{R}} - \frac{\hbar^2}{2\mu}\nabla^2_{\boldsymbol{r}} + \Phi_{ei}(\boldsymbol{r})\right]\psi_{\alpha\boldsymbol{P}}(\boldsymbol{r},\boldsymbol{R}) = E_{\alpha\boldsymbol{P}}\Psi_{\alpha\boldsymbol{P}}(\boldsymbol{r},\boldsymbol{R}) . \qquad (79.6)$$

It follows from this equation that

$$\Psi_{\alpha\boldsymbol{P}}(\boldsymbol{r},\boldsymbol{R}) = \Psi_a(\boldsymbol{r})\,\Psi_{\boldsymbol{P}}(\boldsymbol{R}), \qquad E_{\alpha\boldsymbol{P}} = E_\alpha + E_{\boldsymbol{P}} . \qquad (79.7)$$

The subscript α denotes the quantum numbers $n, \boldsymbol{p}$ (n for the discrete spectrum and $\boldsymbol{p}$ for the continuous spectrum). The normalization condition for the eigenfunctions is

$$\Psi_{\boldsymbol{P}} = V^{-\frac{1}{2}}\exp\left(i\hbar^{-1}\,\boldsymbol{P}\cdot\boldsymbol{R}\right)$$

$$\int d\boldsymbol{R}\,\Psi_{\boldsymbol{P}'}(\boldsymbol{R})\,\Psi^*_{\boldsymbol{P}''}(\boldsymbol{R}) = \frac{(2\pi\hbar)^3}{V}\,\delta(\boldsymbol{P}'-\boldsymbol{P}'')$$

$$\int d\boldsymbol{r}\,\Psi_\alpha(\boldsymbol{r})\,\Psi^*_\beta(\boldsymbol{r}) = \delta_{\alpha\beta} = \begin{cases} \delta_{nm}\,, & \text{for } \alpha=n\,,\ \beta=m \\ 0\,, & \text{for } \begin{cases}\alpha=\boldsymbol{p}'\,, & \beta=m \\ \alpha=n\,, & \beta=\boldsymbol{p}''\end{cases} \\ \dfrac{(2\pi\hbar)^3}{V}\,\delta(\boldsymbol{p}'-\boldsymbol{p}'')\,, & \text{for } \alpha=\boldsymbol{p}'\,,\ \beta=\boldsymbol{p}'' . \end{cases} \qquad (79.8)$$

We use the following representation in terms of the eigenfunctions

$$\hat{\rho}_{ei} = \frac{V^2}{(2\pi\hbar)^6}\sum_\alpha\sum_\beta d\boldsymbol{P}'\,d\boldsymbol{P}''\,\hat{\rho}_{\alpha\beta}(\boldsymbol{P}',\boldsymbol{P}'',t)$$
$$\cdot\,\psi_a(\boldsymbol{r}')\,\Psi^*_\beta(\boldsymbol{r}'')\,\Psi_{\boldsymbol{P}'}(\boldsymbol{R}')\,\Psi^*_{\boldsymbol{P}''}(\boldsymbol{R}'') . \qquad (79.9)$$

From (79.2), (79.9) we find the following equation for the density matrix

$$i\hbar\,\frac{\partial\hat{\rho}_{\alpha\beta}(\boldsymbol{P}',\boldsymbol{P}'',t)}{\partial t} = (E_\alpha + E_{\boldsymbol{P}'} - E_\beta - E_{\boldsymbol{P}''})\,\hat{\rho}_{\alpha\beta}(\boldsymbol{P}',\boldsymbol{P}'',t)$$
$$+\frac{V}{(2\pi\hbar)^3}\sum_\gamma\int d\boldsymbol{P}\left[\hat{U}_{\alpha\gamma}(\boldsymbol{P}',\boldsymbol{P})\,\hat{\rho}_{\gamma\beta}(\boldsymbol{P},\boldsymbol{P}'',t)\right.$$
$$\left.-\,\hat{\rho}_{\alpha\gamma}(\boldsymbol{P}',\boldsymbol{P},t)\,\hat{U}_{\gamma\beta}(\boldsymbol{P},\boldsymbol{P}'')\right] \qquad (79.10)$$

where

$$\sum_{\gamma} \ldots = \sum_{\alpha} \ldots + \frac{V}{(2\pi\hbar)^3} \int d\boldsymbol{p} \ \ldots \ , \quad \alpha = n, \boldsymbol{p}' \ , \quad \beta = m, \boldsymbol{p}'' \tag{79.11}$$

and

$$\hat{U}_{\alpha\beta}(\boldsymbol{P}', \boldsymbol{P}'', t) = \int d\boldsymbol{r}\, d\boldsymbol{R}\, \Psi^{\star}_{\alpha}(\boldsymbol{r})\, \Psi^{\star}_{\boldsymbol{P}'}(\boldsymbol{R})\, \hat{U}_{ei}(\boldsymbol{R}, \boldsymbol{r}, t)\, \Psi_{\boldsymbol{P}''}(\boldsymbol{R})\, \Psi_{\beta}(\boldsymbol{r}) \,. \tag{79.12}$$

Equation (79.10) is shorthand for a set of four equations for the functions $\hat{\rho}_{nm}(\boldsymbol{P}', \boldsymbol{P}'', t)$, $\hat{\rho}_{\boldsymbol{p}', \boldsymbol{p}''}$, $\hat{\rho}_{n, \boldsymbol{p}'}$, $\hat{\rho}_{\boldsymbol{p}', m}$. The first two equations describe the buund and the free states of the particles, coupled in the atoms, whereas the last two equations describe the transitions between bound and free states.

The set of equations (79.10), (79.12) and (79.4) are the starting point for the derivation of the kinetic equations for partially ionized plasmas.

80. KINETIC EQUATIONS FOR THE DISTRIBUTION FUNCTIONS OF ELECTRONS, IONS AND ATOMS

We shall derive these equations in two steps. We first obtain the kinetic equation for the atom distribution function under the assumption that the degree of ionization is zero. Subsequently, we introduce the condition of weakening of the correlations, corresponding to the formation of free charged particles by the ionization of the atoms.

We consider a spatially homogeneous gas, thus:

$$\left\langle \hat{\rho}_{\alpha\beta}(\boldsymbol{P}', \boldsymbol{P}'', t) \right\rangle = \delta_{\alpha\beta} \frac{(2\pi\hbar)^3}{V} \delta(\boldsymbol{P}' - \boldsymbol{P}'')\, f_{\alpha}(\boldsymbol{P}', t)$$

$$\left\langle \hat{U} \right\rangle = 0 \tag{80.1}$$

and the equation for the distribution function f_{α} is (see (74.6)) :

$$\frac{\partial f_{\alpha}(\boldsymbol{P}', t)}{\partial t} = - \frac{2V}{(2\pi\hbar)^3} \sum_{\beta} \int d\boldsymbol{P}'' \ \mathrm{Im} \left\langle \delta\rho_{\alpha\beta}(\boldsymbol{P}', \boldsymbol{P}'', t)\, \delta U_{\alpha\beta}(\boldsymbol{P}', \boldsymbol{P}'', t) \right\rangle \equiv I_{\alpha}(\boldsymbol{P}', t) \tag{80.2}$$

If the correlation $\langle \delta\rho_{\alpha\beta}\, \delta U_{\alpha\beta} \rangle$ is evaluated in the polarization approximation (see sects 73, 74), we find for the collision integral:

$$\begin{aligned} I_{\alpha}(\boldsymbol{P}', t) = \frac{1}{(2\pi)^3 \hbar} \sum_{\beta} \int & d\omega\, d\boldsymbol{k}\, d\boldsymbol{P}'' \frac{|P_{\alpha\beta}(\boldsymbol{k})|^2}{k^2} \delta(\hbar\boldsymbol{k} - \boldsymbol{P}' + \boldsymbol{P}'') \\ & \delta(\hbar\omega - E_{\alpha} - E_{\boldsymbol{P}'} + E_{\beta} + E_{\boldsymbol{P}''}) \\ & \cdot \left\{ \left(\delta \boldsymbol{E} \cdot \delta \boldsymbol{E}\right)_{\omega \boldsymbol{k}} \left[f_{\beta}(\boldsymbol{P}'', t) - f_{\alpha}(\boldsymbol{P}', t) \right] \right. \\ & \left. - \frac{4\pi\, \mathrm{Im}\, \varepsilon(\omega, \boldsymbol{k})}{|\varepsilon(\omega, \boldsymbol{k})|^2} \left[f_{\beta}(\boldsymbol{P}'', t) + f_{\alpha}(\boldsymbol{P}', t) \right] \right\} \end{aligned} \tag{80.3}$$

The dielectric constant has the following form

$$\varepsilon(\omega,\boldsymbol{k}) = 1 + \frac{4\pi n}{k^2}\,\frac{V}{(2\pi\hbar)^3}\sum_\alpha\sum_\beta\int d\boldsymbol{P}'\,d\boldsymbol{P}''\,\delta(\hbar\boldsymbol{k}-\boldsymbol{P}'+\boldsymbol{P}'')$$

$$\frac{|P_{\alpha\beta}(\boldsymbol{k})|^2\left[f_\alpha(\boldsymbol{P}',t)-f_\beta(\boldsymbol{P}'',t)\right]}{\hbar(\omega+i\Delta)-(E_\alpha+E_{\boldsymbol{P}'}-E_\beta-E_{\boldsymbol{P}''})}\;. \tag{80.4}$$

The spectral density of the field fluctuations is :

$$\left(\delta\boldsymbol{E}\cdot\delta\boldsymbol{E}\right)_{\omega,\boldsymbol{k}} = \frac{2nV}{\hbar^2}\sum_\alpha\sum_\beta\int d\boldsymbol{P}'\,d\boldsymbol{P}''\left[f_\alpha(\boldsymbol{P}',t)+f_\beta(\boldsymbol{P}'',t)\right]$$

$$\frac{|P_{\alpha\beta}(\boldsymbol{k})|^2}{k^2\,|\varepsilon(\omega,\boldsymbol{k})|^2}\,\delta(\hbar\boldsymbol{k}-\boldsymbol{P}'+\boldsymbol{P}'')\,\delta(\hbar\omega-E_\alpha-E_{\boldsymbol{P}'}+E_\beta+E_{\boldsymbol{P}''})\;. \tag{80.5}$$

Finally, the matrix element entering here is

$$P_{\alpha\beta}(\boldsymbol{k}) = \int d\boldsymbol{r}\;\Psi^*_\alpha(\boldsymbol{r})\,\Psi_\beta(\boldsymbol{r})$$

$$\cdot\left[e_e\exp\left(i\frac{m_i}{m_e+m_i}\,\boldsymbol{k}\cdot\boldsymbol{r}\right)+e_i\exp\left(-i\frac{m_e}{m_e+m_i}\,\boldsymbol{k}\cdot\boldsymbol{r}\right)\right]\;. \tag{80.6}$$

Equations (80.3) – (80.5) correspond to Eqs. (74.7), (74.2), (74.1) for the fully ionized plasma. If the functions $(\delta\boldsymbol{E}\cdot\delta\boldsymbol{E})_{\omega,\boldsymbol{k}}$ and $\mathrm{Im}\,\varepsilon(\omega,\boldsymbol{k})$ are eliminated by using (80.4), (80.5), we get an expression analogous to (74.9) or (74.10).

In order to obtain from (80.3) the collision integrals for electrons, ions and atoms we proceed as follows. We noted already that by analogy to (79.11) the right-hand side of (80.3) can be represented as a sum of two terms, taking into account the contributions of the discrete and of the continuous spectrum. In a similar way, the spectral density $(\delta\boldsymbol{E}\cdot\delta\boldsymbol{E})_{\omega,\boldsymbol{k}}$ and the function $\mathrm{Im}\,\varepsilon(\omega,\boldsymbol{k})$ can be represented as

$$\alpha(\omega,\boldsymbol{k}) \equiv \frac{1}{4\pi}\left[\varepsilon(\omega,\boldsymbol{k})-1\right] = \alpha_{ff}+\alpha_{fb}+\alpha_{bf}+\alpha_{bb}. \tag{80.7}$$

The indices f and b denote, respectively, free and bound states.

For the particles in free states the eigenfunctions are :

$$\Psi_\alpha(\boldsymbol{r}) \longrightarrow \Psi_{\boldsymbol{p}}(\boldsymbol{r}) = V^{-\frac{1}{2}}\exp\left(i\hbar^{-1}\,\boldsymbol{p}\cdot\boldsymbol{r}\right)\;. \tag{80.8}$$

For the description of the free particles it is convenient to use, instead of $\boldsymbol{p},\boldsymbol{P}$ the variables $\boldsymbol{p}_a$, $\boldsymbol{p}_b$ defined as follows

$$\boldsymbol{P} = \boldsymbol{p}_e+\boldsymbol{p}_i \qquad \boldsymbol{p}_e = \frac{m_e}{m_e+m_i}\,\boldsymbol{P}+\boldsymbol{p}\;,$$

$$\boldsymbol{P} = \frac{m_i\boldsymbol{p}_e-m_e\boldsymbol{p}_i}{m_e+m_i} \qquad \boldsymbol{p}_i = \frac{m_i}{m_e+m_i}\,\boldsymbol{P}-\boldsymbol{p}\;. \tag{80.9}$$

Thus

$$f_{\boldsymbol{p}}(\boldsymbol{P},t) \longrightarrow f(\boldsymbol{p}_e,\boldsymbol{p}_i,t)\;. \tag{80.10}$$

The function

$$Nf_{\boldsymbol{p}}(\boldsymbol{P},t) = Nf(\boldsymbol{p}_e,\boldsymbol{p}_i,t) \tag{80.11}$$

defines the mean number of pairs of particles (atoms) in which the particles are so far apart that they can be considered free. If the degree of ionization is zero, such states are impossible, and the collision integral is defined by (80.3), with $\alpha = n$, $\beta = m$.

If the degree of ionization is not zero, the partially ionized gas can be described again by Eq. (80.3) in which we perform the substitution

$$Nf(\boldsymbol{p}_e,\boldsymbol{p}_i,t) \rightarrow Nf(\boldsymbol{p}_e,t)\ Nf(\boldsymbol{p}_i,t). \tag{80.12}$$

This means that we replace the number of pairs of free particles by the product of the average numbers of free particles with the same momentum. This is precisely the assumption of weakening of the correlations, mentioned above.

In order to derive the set of equations for the distribution functions of the electrons, the ions and the atoms we use Eqs (80.2). (80.3) and (80.12). We substitute into (80.2) $\alpha = \boldsymbol{p}'$, perform the change of variables (80.9) and substitute :

$$I_{\boldsymbol{p}'}(\boldsymbol{P}',t) \rightarrow I(\boldsymbol{p}_e,\boldsymbol{p}_i,t) . \tag{80.13}$$

We introduce the following definition for the distribution function of the ions (electrons) :

$$f_a(\boldsymbol{p}_a,t) = \frac{V}{(2\pi\hbar)^3}\int d\boldsymbol{p}_b\ f(\boldsymbol{p}_a,\boldsymbol{p}_b,t) \qquad (b \neq a\ ,\ a = e,i). \tag{80.14}$$

The equations can now be written in the form

$$\frac{\partial f_a(\boldsymbol{p}_a,t)}{\partial t} = \frac{V}{(2\pi\hbar)^3}\int d\boldsymbol{p}_b\ I(\boldsymbol{p}_a,\boldsymbol{p}_b,t) = I_a(\boldsymbol{p}_a,t) . \tag{80.15}$$

Introducing into (80.2), (80.3) $\alpha = n$ we find the kinetic equations for the atoms:

$$\frac{\partial f_n(\boldsymbol{P}',t)}{\partial t} = I_n(\boldsymbol{P}',t) . \tag{80.16}$$

The distribution functions for the electrons, ions and atoms are normalized as follows

$$\frac{V}{(2\pi\hbar)^3}\int d\boldsymbol{p}_a\ f_a(\boldsymbol{p}_a,t) = \frac{N_a}{N} = c_a$$

$$\frac{V}{(2\pi\hbar)^3}\sum_n \int d\boldsymbol{P}\ f_n(\boldsymbol{P},t) = \frac{N_{ei}}{N} = c_{ei} \tag{80.17}$$

where c_a $(a = e,i)$ is the concentration of the electrons and of the ions, and c_{ei} the concentration of the atoms. These concentrations are related by :

$$\sum_a c_a + c_{ei} = 1. \tag{80.18}$$

We now study some properties of these collision integrals. We represent

the integral $I_a(\boldsymbol{p}_a, t)$ as a sum of two terms :

$$I_a(\boldsymbol{p}_a, t) = \Big[I_a(\boldsymbol{p}_a, t)\Big]_1 + \Big[I_a(\boldsymbol{p}_a, t)\Big]_2 \qquad (a = e, i) \tag{80.19}$$

where

$$\begin{aligned}\Big[I_a(\boldsymbol{p}'_\alpha, t)\Big]_1 = \frac{e_a^2}{(2\pi)^3\hbar} \int d\boldsymbol{p}''\, d\omega\, d\boldsymbol{k}\, k^{-2}\, \delta\Big(\hbar \boldsymbol{k} - \boldsymbol{p}'_a + \boldsymbol{p}''_a\Big)\, \delta\Big(\hbar\omega - \frac{p'^2_a - p''^2_a}{2m_a}\Big) \\ \Bigg\{\Big(\delta E \cdot \delta \boldsymbol{E}\Big)_{\omega \boldsymbol{k}} \Big[f_a(\boldsymbol{p}''_a, t) - f_a(\boldsymbol{p}'_a, t)\Big] \\ - \frac{4\pi\hbar\ \mathrm{Im}\,\varepsilon(\omega, \boldsymbol{k})}{|\varepsilon(\omega, \boldsymbol{k})|^2} \Big[f_a(\boldsymbol{p}''_a, t) + f_a(\boldsymbol{p}'_a, t)\Big]\Bigg\}\ .\end{aligned} \tag{80.20}$$

In order to obtain this expression, we put $\alpha = \boldsymbol{p}'$, $\beta = \boldsymbol{p}''$ in (80.3), use (80.9) and express the matrix element (80.6) in terms of free-motion eigenfunctions.

As the spectral density of the field fluctuations and the imaginary part of the dielectric constant consist of a sum of four terms, Eq. (80.20) also splits up into four collision integrals, describing four different processes :

1. $\boldsymbol{p}'_a + \boldsymbol{p}''_{1b} \longleftrightarrow \boldsymbol{p}''_a + \boldsymbol{p}'_{1b}$

is the scattering of free particles;

2. $\boldsymbol{p}'_a + m_1 \boldsymbol{P}''_1 \longleftrightarrow \boldsymbol{p}''_a + \boldsymbol{p}'_{1a} + \boldsymbol{p}'_{1b} \qquad (a \neq b)$

is the ionization of an atom in the state $m_1\, \boldsymbol{P}''_1$ through the collision with an electron or an ion;

3. $\boldsymbol{p}'_a + \boldsymbol{p}''_{1a} + \boldsymbol{p}''_{1b} \longleftrightarrow \boldsymbol{p}''_a + n_1\, \boldsymbol{P}'_1$

represents the recombination in presence of an electron or of an ion;

4. $\boldsymbol{p}'_a + m_1\, \boldsymbol{P}''_1 \longleftrightarrow \boldsymbol{p}''_a + n_1\, \boldsymbol{P}'_1$

represents an inelastic collision of an ion or electron with an atom.

In all these four processes, the average number of particles of type a, interacting with free particles or with atoms, does not change; thus

$$\frac{V}{(2\pi\hbar)^3} \int d\boldsymbol{p}_a\, \Big[I_a(\boldsymbol{p}_a, t)\Big]_1 = 0 \tag{80.21}$$

and the concentrations of the free particles, c_a, does not change in these processes.

The second term on the right-hand side of (80.19) is defined by :

$$\left[I_a(\boldsymbol{p}_a\,,t)\right]_2 = \frac{1}{(2\pi)^3\hbar}\,\frac{V}{(2\pi\hbar)^3}\sum_m \int d\boldsymbol{p}'_b\, d\boldsymbol{P}''\, d\omega\, d\boldsymbol{k}\; k^{-2}$$

$$\cdot \left| P_{(m_b \boldsymbol{p}'_a - m_a \boldsymbol{p}'_b)/(m_a+m_b)\,,m}\right|^2 \;\delta(\hbar\boldsymbol{k} - \boldsymbol{p}_a - \boldsymbol{p}'_b + \boldsymbol{P}'')$$

$$\delta\left(\hbar\omega - \frac{p'^2_a}{2m_a} - \frac{p'^2_b}{2m_b} + E_m + E_{\boldsymbol{P}'}\right)$$

$$\cdot \left\{ \left(\delta\boldsymbol{E}\cdot\delta\boldsymbol{E}\right)_{\omega,\boldsymbol{k}} \left[f_m(\boldsymbol{P}'',t) - N f_a(\boldsymbol{p}'_a,t)\, f_b(\boldsymbol{p}'_b,t)\right]\right.$$

$$\left. - \frac{4\pi\hbar\, \mathrm{Im}\,\varepsilon(\omega,\boldsymbol{k})}{|\varepsilon(\omega,\boldsymbol{k})|^2}\left[f_m(\boldsymbol{P}'',t) + N f_a(\boldsymbol{p}'_a\,,t)\;\; f_b(\boldsymbol{p}',t)\right]\right\} . \quad (b \neq a) \qquad (80.22)$$

In order to obtain this equation, we set in (80.3) $\alpha = \boldsymbol{p}'$, $\beta = \mathrm{m}$, use (80.9), integrate over $\boldsymbol{p}'_b$ and use (80.12) under the integral. We again distinguish four processes.

5. $$\boldsymbol{p}'_a + \boldsymbol{p}'_b + \boldsymbol{p}''_{1a} \leftrightarrow m\,\boldsymbol{P}'' + \boldsymbol{p}'_{1a}$$

is the recombination of particles a and b $(a \neq b)$;

6. $$\boldsymbol{p}'_a + \boldsymbol{p}'_b + m_1\,\boldsymbol{P}''_1 \leftrightarrow m\,\boldsymbol{P}'' + \boldsymbol{p}'_{1a} + \boldsymbol{p}'_{1b}$$

is a process of inelastic scattering, accompanied by an exchange of particles;

7. $$\boldsymbol{p}'_a + \boldsymbol{p}'_b + \boldsymbol{p}''_{1a} + \boldsymbol{p}''_{1b} \leftrightarrow m\,\boldsymbol{P}'' + n_1\,\boldsymbol{P}'_1$$

is the process of double recombination;

8. $$\boldsymbol{p}'_a + \boldsymbol{p}'_b + m_1\,\boldsymbol{P}''_1 \leftrightarrow m\,\boldsymbol{P}'' + n_1\,\boldsymbol{P}'_1$$

is a recombination process in the presence of an atom.

For the collision integral (80.22) the property (80.21) does not hold :

$$\frac{V}{(2\pi\hbar)^3}\int d\boldsymbol{p}_a \left[I_a(\boldsymbol{p}_a,t)\right]_2 \neq 0 \; . \qquad (80.23)$$

Thus, in the processes 5—8 the concentrations of the free particles change.

If the plasma is completely ionized, only the process (1) is possible. Thus, instead of (80.7), we have

$$\varepsilon(\omega,\boldsymbol{k}) = 1 + 4\pi\alpha_{ff} \qquad (80.24)$$

and the dielectric constant reduces to (74.2), whereas (80.5) reduces to (71.1). The collision integral (80.22) vanishes, and (80.20) reduces to the form (74.7) of the quantum Balescu-Lenard collision integral.

The collision integral for the atoms can be represented as

$$I_n(\boldsymbol{P}',t) = \left[I_n(\boldsymbol{P}',t)\right]_1 + \left[I_n(\boldsymbol{P}',t)\right]_2 . \qquad (80.25)$$

The first term follows from (80.1) with $\alpha = n$, $\beta = m$. We have here the following four processes :

1'. $$n\boldsymbol{P}' + \boldsymbol{p}''_{1b} \longleftrightarrow m\boldsymbol{P}'' + \boldsymbol{p}'_{1b}$$

2'. $$n\boldsymbol{P}' + m_1\boldsymbol{P}''_1 \longleftrightarrow m\boldsymbol{P}'' + \boldsymbol{p}'_{1a} + \boldsymbol{p}'_{1b}$$

3'. $$n\boldsymbol{P}' + \boldsymbol{p}''_{1a} + \boldsymbol{p}''_{1b} \longleftrightarrow m\boldsymbol{P}'' + n_1\boldsymbol{P}'_1$$

4'. $$n\boldsymbol{P}' + m_1\boldsymbol{P}''_1 \longleftrightarrow m\boldsymbol{P}'' + n_1\boldsymbol{P}'_1$$

These processes do not modify the number of atoms, thus

$$\sum_n \int d\boldsymbol{P}' \left[I_n(\boldsymbol{P}', t) \right]_1 = 0 . \tag{80.26}$$

In order to obtain the second part of the collision integral, we set in (80.3) $\alpha = n$, $\beta = \boldsymbol{p}''$, use (80.9) and (80.12) and obtain

$$\left[I_n(\boldsymbol{P}', t) \right]_2 = \frac{1}{(2\pi)^3 \hbar} \frac{V}{(2\pi\hbar)^3}$$
$$\int d\boldsymbol{p}''_a \, d\boldsymbol{p}''_b \, d\omega \, d\boldsymbol{k} \; k^{-2} \left| P_{n, (m_b \boldsymbol{p}''_a - m_a \boldsymbol{p}''_b)/(m_a + m_b)} \right|$$
$$\delta(\hbar\boldsymbol{k} - \boldsymbol{P}' + \boldsymbol{p}''_a + \boldsymbol{p}''_b) \; \delta\left(\hbar\omega - E_n - E_{\boldsymbol{P}'} + \frac{p''^2_a}{2m_a} + \frac{p''^2_b}{2m_b} \right)$$
$$\cdot \left\{ \left(\delta\boldsymbol{E} \cdot \delta\boldsymbol{E} \right)_{\omega, \boldsymbol{k}} \left[N f_a(\boldsymbol{p}''_a) f_b(\boldsymbol{p}''_b) - f_n(\boldsymbol{P}') \right] \right.$$
$$\left. - \frac{4\pi\hbar \, \mathrm{Im}\, \varepsilon(\omega, \boldsymbol{k})}{|\varepsilon(\omega, \boldsymbol{k})|^2} \left[N f_a(\boldsymbol{p}''_a) f_b(\boldsymbol{p}''_b) + f_n(\boldsymbol{P}') \right] \right\} \tag{80.27}$$

Here we find the following four processes :

5'. $$n\boldsymbol{P}' + \boldsymbol{p}''_{1a} \longleftrightarrow \boldsymbol{p}''_a + \boldsymbol{p}''_b + \boldsymbol{p}'_{1a}$$

6'. $$n\boldsymbol{P}' + \boldsymbol{p}''_{1a} + \boldsymbol{p}''_{1b} \longleftrightarrow \boldsymbol{p}''_a + \boldsymbol{p}''_b + n_1\boldsymbol{P}'_1$$

7'. $$n\boldsymbol{P}' + m_1\boldsymbol{P}''_1 \longleftrightarrow \boldsymbol{p}''_a + \boldsymbol{p}''_b + \boldsymbol{p}'_{1a} + \boldsymbol{p}'_{1b}$$

8'. $$n\boldsymbol{P}' + m_1\boldsymbol{P}''_1 \longleftrightarrow \boldsymbol{p}''_a + \boldsymbol{p}''_b + n_1\boldsymbol{P}'_1$$

These processes change the number of atoms, thus :

$$\sum_n \int d\boldsymbol{P}' \frac{V}{(2\pi\hbar)^3} \left[I_n(\boldsymbol{P}', t) \right]_2 \neq 0 . \tag{80.28}$$

The collision integrals (80.19), (80.25) possess the following conservation properties :

$$I(t) = \frac{N}{V} \sum_\alpha \int d\boldsymbol{P}' \frac{V}{(2\pi\hbar)^3} \phi_\alpha(\boldsymbol{P}') \, I_\alpha(\boldsymbol{P}', t) = 0 , \tag{80.29}$$

for $\phi_\alpha(\boldsymbol{P}') = 1$, $\boldsymbol{P}'$, $E_\alpha + E_{\boldsymbol{p}'}$. These properties express the conservation of the total number of charged particles of each type (in free and bound states), the total momentum and the total energy of the free particles and of the atoms. Furthermore:

$$I(t) \geqslant 0 \quad \text{for} \quad \phi_\alpha = -k_\mathrm{B} \ln f_\alpha(\boldsymbol{P}', t) \ . \tag{80.30}$$

This expresses the increase of the total entropy for the isolated partially ionized plasma. In order to prove these relations, we need Eq. (79.11).

We consider the equilibrium solutions of (80.15), (80.16), determined by the equations

$$\begin{aligned} I(\boldsymbol{p}_a, t) &= \Big[I(\boldsymbol{p}_a, t)\Big]_1 + \Big[I(\boldsymbol{p}_a, t)\Big]_2 = 0 \ , \\ I_n(\boldsymbol{P}, t) &= \Big[I_n(\boldsymbol{P}, t)\Big]_1 + \Big[I_n(\boldsymbol{P}, t)\Big]_2 = 0 \ . \end{aligned} \tag{80.31}$$

In order to satisfy these equations it is not sufficient to have the equilibrium distributions for the electrons, ions and atoms :

$$f(\boldsymbol{p}_a) = \frac{1}{N} \exp\left[\frac{\mu_a - E_{\boldsymbol{p}_a}}{k_\mathrm{B} T}\right] , \qquad f_n(\boldsymbol{P}) = \frac{1}{N} \exp\left[\frac{\mu_{ei} - E_n - E_{\boldsymbol{p}}}{k_\mathrm{B} T}\right] \tag{80.32}$$

with

$$\mu_a = k_\mathrm{B} T \ln\left[\frac{N_a}{V}\left(\frac{2\pi\hbar^2}{m_a k_\mathrm{B} T}\right)^{\frac{1}{2}}\right] , \quad \mu_{ei} = k_\mathrm{B} T \ln\left[\frac{N_{ei}}{V}\left(\frac{2\pi\hbar^2}{M k_\mathrm{B} T}\right)^{\frac{1}{2}} \cdot \frac{1}{Z}\right] \tag{80.33}$$

(Z is the partition function). We must also impose the relation

$$\mu_e + \mu_i = \mu_{ei} \tag{80.34}$$

among the chemical potentials, which expresses the condition of chemical equilibrium. The latter can also be writen as follows :

$$\frac{c_e c_i}{c_{ei}} \equiv \frac{n_e n_i}{n_{ei}} = \left(\frac{\mu_{ei} k_\mathrm{B} T}{2\pi\hbar^2}\right)^{\frac{1}{2}} \frac{1}{Z} \ . \tag{80.35}$$

This is Saha's formula. Thus, in equilibrium, the distributions of the electrons, ions and atoms are Maxwell-Boltzmann distributions, related through the condition of chemical equilibrium.

From the kinetic equations (80.15),(80.16) we may obtain the hydrodynamical equations for a partially ionized gas, taking into account the chemical transformation [22] .

In reference [37] a similar method was used for the derivation of the kinetic equations of a system of charged particles with electromagnetic interactions.

81. CHEMICALLY REACTING GASES. INITIAL MODEL

We consider a three-component gas, consisting of atoms a and b, and of molecules ab. We may again consider two limiting cases: the molecular gas (degree of dissociation equal to zero), and the fully dissociated gas, a mixture of two atomic gases.

We take as a reference state, as in section 79, the molecular gas. We assume the gas to be sufficienly dilute, in order to be allowed to use the binary collision approximation. The state of the gas is then described by the two-particle density matrix

$$\rho_{a_1 b_1 a_2 b_2} = \rho_{a_1 b_1 a_2 b_2}(\mathbf{r}'_{a_1}, \mathbf{r}'_{b_1}, \mathbf{r}'_{a_2}, \mathbf{r}'_{b_2}, \mathbf{r}''_{a_1}, \mathbf{r}''_{b_1}, \mathbf{r}''_{a_2}, \mathbf{r}''_{b_2}, t) \tag{81.1}$$

where 1 and 2 are the molecules, and a, b are the subscripts of the atoms in the molecules.

Let

$$\hat{H}_{ab} = -\frac{\hbar^2}{2m_a}\nabla^2_{\mathbf{r}_a} - \frac{\hbar^2}{2m_b}\nabla^2_{\mathbf{r}_b} + \Phi_{ab} \tag{81.2}$$

be the Hamiltonian of a molecule,

$$\hat{H}^0_{a_1 b_1 a_2 b_2} = \hat{H}_{a_1 b_1} + \hat{H}_{a_2 b_2} \tag{81.3}$$

be the Hamiltonian of two non-interacting atoms, and

$$U_{a_1 b_1 a_2 b_2} = U(\mathbf{r}_{a_1}, \mathbf{r}_{b_1}, \mathbf{r}_{a_2}, \mathbf{r}_{b_2}) \tag{81.4}$$

be the interaction potential of two molecules. If both interacting molecules are in the dissociated state, then

$$U_{a_1 b_1 a_2 b_2} = \Phi_{a_1 a_2} + \Phi_{a_1 b_2} + \Phi_{b_1 a_2} + \Phi_{b_1 b_2} \tag{81.5}$$

where Φ_{ab} is the interaction potential of the atoms a, b. If only one molecule, say 2, is dissociated, then

$$U_{a_1 b_1 a_2 b_2} = V_{a_1 b_1 a_2} + V_{a_1 b_1 b_2}. \tag{81.6}$$

Thus, the potential U is additive only in the case of free atoms. In the general case we have to know the following potentials :

$$U_{a_1 b_1 a_2 b_2}\,,\quad V_{a_1 b_1 a_2}\,,\quad V_{a_1 b_1 b_2}\,,\quad \Phi_{aa}\,,\quad \Phi_{ab}\,,\quad \Phi_{bb} \tag{81.7}$$

The equations for the density matrices in the binary collision approximation can be written by analogy with (66.10) as follows :

$$\left[i\hbar \frac{\partial}{\partial t} - \hat{H}^0_{a_1 b_1 a_2 b_2}\left(\boldsymbol{r}'_{a_1}, \boldsymbol{r}'_{b_1}, \boldsymbol{r}'_{a_2}, \boldsymbol{r}'_{b_2}\right) + \hat{H}^0_{a_1 b_1 a_2 b_2}\left(\boldsymbol{r}''_{a_1}, \boldsymbol{r}''_{b_1}, \boldsymbol{r}''_{a_2}, \boldsymbol{r}''_{b_2}\right)\right] \cdot \left(\rho_{a_1 b_1 a_2 b_2} - \rho_{a_1 b_1}\rho_{a_2 b_2}\right) = \left[U_{a_1 b_1 a_2 b_2}\left(\boldsymbol{r}'_{a_1}, \boldsymbol{r}'_{b_1}, \boldsymbol{r}'_{a_2}, \boldsymbol{r}'_{b_2}\right) - U_{a_1 b_1 a_2 b_2}\left(\boldsymbol{r}''_{a_1}, \boldsymbol{r}''_{b_1}, \boldsymbol{r}''_{a_2}, \boldsymbol{r}''_{b_2}\right)\right] \rho_{a_1 b_1 a_2 b_2} . \tag{81.8}$$

In the weak coupling approximation for the molecular interactions, we may take in the right-hand side :

$$\rho_{a_1 b_1 a_2 b_2} = \rho_{a_1 b_1}\rho_{a_2 b_2} \tag{81.9}$$

and find an equation analogous to (69.1).

82. KINETIC EQUATIONS FOR CHEMICALLY REACTING GASES

The derivation of the kinetic equations can again be done in two steps, as in section 79. First one derives kinetic equations for the two-molecule distribution functions, then one identifies three kinetic equations for the atoms and molecules, by using the assumption of weakening of the correlations in the formation of free atoms.

The kinetic equations were derived in ref. [38] (see also [45,46]). We only quote a few results here.

We assume that the weak-coupling approximation is valid. For a spatially homogeneous gas in the diagonal approximation, the kinetic equation for the distribution function $f_{n_{a_1 b_1}}(\boldsymbol{P}'_{a_1 b_1}, t)$ can be written as follows :

$$\frac{\partial f_{n_{a_1 b_1}}}{\partial t} = I_{n_{a_1 b_1}}\left(\boldsymbol{P}'_{a_1 b_1}, t\right). \tag{82.1}$$

The collision integral is defined as follows :

$$I_{n_{a_1 b_1}}\left(\boldsymbol{P}'_{a_1 b_1}, t\right) = \frac{2\pi N}{\hbar(2\pi\hbar)^6} \sum_{m_{a_1 b_1}} \sum_{n_{a_2 b_2}} \sum_{m_{a_2 b_2}} \int d\boldsymbol{P}''_{a_1 b_1}\, d\boldsymbol{P}'_{a_2 b_2}\, d\boldsymbol{P}''_{a_2 b_2} \left|U_{n_{a_1 b_1} m_{a_1 b_1} n_{a_2 b_2} m_{a_2 b_2}}\left(\hbar^{-1}\left|\boldsymbol{P}'_{a_1 b_1} - \boldsymbol{P}''_{a_1 b_1}\right|\right)\right|^2 \tag{82.2}$$

$$\cdot\delta\left(\boldsymbol{P}'_{a_1b_1}+\boldsymbol{P}'_{a_2b_2}-\boldsymbol{P}''_{a_1b_1}-\boldsymbol{P}''_{a_2b_2}\right)$$

$$\delta\left(E_{n_{a_1b_1}}+E_{\boldsymbol{P}'_{a_1b_1}}+E_{n_{a_2b_2}}+E_{\boldsymbol{P}'_{a_2b_2}}-E_{m_{a_1b_1}}-E_{\boldsymbol{P}''_{a_1b_1}}-E_{m_{a_2b_2}}-E_{\boldsymbol{P}''_{a_1b_1}}\right)$$

$$\left[f_{m_{a_1b_1}}\left(\boldsymbol{P}''_{a_1b_1},t\right)f_{m_{a_2b_2}}\left(\boldsymbol{P}''_{a_2b_2},t\right)\right.$$

$$\left.-f_{n_{a_1b_1}}\left(\boldsymbol{P}'_{a_1b_1},t\right)f_{n_{a_2b_2}}\left(\boldsymbol{P}'_{a_2b_2},t\right)\right] \qquad \begin{matrix}(82.2)\\ (\text{cont})\end{matrix}$$

where N is the number of molecules in the zero-dissociation state, and

$$U_{n_{a_1b_1}m_{a_1b_1}n_{a_2b_2}m_{a_2b_2}}(k)=\int d\boldsymbol{r}_{a_1b_1}\,d\boldsymbol{r}_{a_2b_2}\,U\left(\boldsymbol{k},\boldsymbol{r}_{a_1b_1},\boldsymbol{r}_{a_2b_2}\right)$$

$$\Psi^{\star}_{n_{a_1b_1}}\left(\boldsymbol{r}_{a_1b_1}\right)\Psi^{\star}_{n_{a_2b_2}}\left(\boldsymbol{r}_{a_2b_2}\right)\Psi_{m_{a_2b_2}}\left(\boldsymbol{r}_{a_2b_2}\right)\Psi_{m_{a_1b_1}}\left(\boldsymbol{r}_{a_1b_1}\right) \qquad (82.3)$$

is the matrix element of the molecular interaction energy. $\Psi_{n_{ab}}(\boldsymbol{r}_{ab})$ are eigenfunctions of the Hàmiltonian $\hat{H}_{ab}$, and $U\left(\boldsymbol{k},\boldsymbol{r}_{a_1b_1},\boldsymbol{r}_{a_2b_2}\right)$ is the Fourier transform of the interaction energy of the molecules :

$$U_{a_1b_1a_2b_2}=U_{a_1b_1a_2b_2}\left(\boldsymbol{R}_{a_1b_1}-\boldsymbol{R}_{a_2b_2},\boldsymbol{r}_{a_1b_1},\boldsymbol{r}_{a_2b_2}\right)$$

In the elastic collision approximation :

$$U_{n_{a_1b_1}m_{a_1b_1}n_{a_2b_2}m_{a_2b_2}}(\boldsymbol{k})=\nu(\boldsymbol{k})\,\delta_{n_{a_1b_1}m_{a_1b_1}}\,\delta_{n_{a_2b_2}m_{a_2b_2}} \qquad (82.4)$$

and from (82.1) follows the relation

$$I(\boldsymbol{P}_{ab},t)=\sum_{n_{ab}}I_{n_{ab}}(\boldsymbol{P}_{ab},t).$$

For monatomic gases this relation reduces to the Bogolyubov-Gurov collision integral (69.11).

From (82.2) we may derive also the collision integral for particles with Coulomb interactions, but without polarization effects, because we used the weak-coupling approximation to start. In that case we use (81.5) for U and obtain for the matrix element (82.3) :

$$\left|U_{n_{a_1b_1}m_{a_1b_1}n_{a_2b_2}m_{a_2b_2}}(\boldsymbol{k})\right|$$

$$=\left(\frac{4\pi}{k^2}\right)^2\left|P_{n_{a_1b_1}m_{a_1b_1}}(\boldsymbol{k})\right|^2\left|P_{n_{a_2b_2}m_{a_2b_2}}(\boldsymbol{k})\right|^2. \qquad (82.5)$$

We used here the expression (80.6) of the matrix element.

Using a relation analogous to (80.12) we may derive, as in section 80, a set of three equations for the distribution functions of the atoms and of the

molecules. These equations have the same structure as (80.15), (80.16) (see ref. [38]).

83. EQUATIONS FOR THE CONCENTRATIONS OF FREE AND BOUND CHARGED PARTICLES. IONIZATION AND RECOMBINATION COEFFICIENTS

We consider a non-equilibrium state of a homogeneous partially ionized plasma. We assume that the Maxwell-Boltzmann state is already reached for the electrons, the ions and the atoms, but that the state of chemical (ionization) equilibrium is not yet attained. As a result, the concentrations of the components are not related by the ionization equilibrium relation (80.35).

In order to obtain an equation for $n_a(t)$, say, we multiply the kinetic equation (80.15) by $nV/(2\pi\hbar)^3$, integrate over $\boldsymbol{p}_a$, and use (80.5), (80.4), (80.7) and (80.32). We obtain [22] :

$$\frac{dn_a(t)}{dt} = \left(\alpha\, n_a n_{ab} - \beta n_a^3\right) + \left(\alpha_1 n_{ab}^2 - \beta_1 n_a^2 n_{ab}\right) + \left(\alpha_2 n_{ab}^2 - \beta_2 n_a^2 n_b^2\right) + \left(\alpha_3 n_a^2 n_{ab} - \beta_3 n_a^2 n_{ab}\right). \tag{83.1}$$

We used here the equality of the concentration of ions and electrons. α is the coefficient of collisional ionization, β the coefficient of triple recombination (two electrons and an ion, or two ions and an electron), α_1 the coefficient of ionization through the collision of two atoms, β_1 the coefficient of recombination through triple collisions of an ion, an electron and an atom, β_2 the coefficient of recombination of four particles, yielding two atoms, α_2 the corresponding ionization coefficient, α_3 and β_3 coefficients of exchange processes.

As an example we consider the coefficients α and β [22]:

$$\alpha = \frac{4V}{(2\pi\hbar)^3} \sum_c e_c^2 \sum_m \int d\boldsymbol{p}'\, d\boldsymbol{P}'\, d\boldsymbol{P}''\, d\boldsymbol{p}_c'\, d\boldsymbol{p}_c''\, d\omega\, d\boldsymbol{k}\, \frac{|P_{\boldsymbol{p}'m}|^2}{k^4 |\varepsilon(\omega,\boldsymbol{k})|^2}$$
$$\delta\left(\boldsymbol{P}' + \boldsymbol{p}_c'' - \boldsymbol{P}'' - \boldsymbol{p}_c'\right) \delta\left(\frac{p'^2}{2\mu} + \frac{P'^2}{2M} + \frac{p_c''^2}{2m_c} - E_m - \frac{P''^2}{2M} - \frac{p_c'^2}{2m_c}\right)$$
$$\delta\left(\hbar\boldsymbol{k} - \boldsymbol{P}' + \boldsymbol{P}''\right) \delta\left(\hbar\omega - \frac{p'^2}{2\mu} - \frac{P'^2}{2M} + E_m + \frac{P''^2}{2M}\right)$$
$$\frac{1}{[2\pi (m_c M)^{\frac{1}{2}} k_{\mathrm{B}}T]^3} \frac{1}{Z} \exp\left[-\left(E_m + \frac{P''^2}{2M} + \frac{p_c'^2}{2m_c}\right)\Big/ k_{\mathrm{B}}T\right] \tag{83.2}$$

$$\beta = \left(\frac{2\pi\hbar^2}{\mu k_{\mathrm{B}}T}\right)^{\frac{3}{2}} Z\alpha\,. \tag{83.3}$$

The integrations in this equation can be performed, if we assume the following simplifications are valid. The polarization effects are unimportant ($\varepsilon(\omega,\boldsymbol{k}) = 1$),

the ionization is mainly due to electrons, the atoms are at rest ($M=\infty$), and the ionization proceeds from the ground state. We then obtain from (83.2) the following expression for the ionization coefficient α:

$$\alpha = \frac{2^{10}}{3^7}\, 35\, e^{-\frac{4}{3}}\, \frac{a_0^3 \mu e^4}{\hbar^3} \frac{k_B T}{I} e^{-I/k_B T} \tag{83.4}$$

where I is the ionization potential, a_0 the Bohr radius. In deriving this formula, we used the expression for the matrix element $|\boldsymbol{r}_{\boldsymbol{p}'m}(\boldsymbol{k})|^2$ for $m=0$ (see reference [40], p.667).

If the ionization coefficient is calculated classically [41] one obtains a result differing from (83.4) in the pre-exponential factor: instead of $(k_B T/I)$, one obtains $(k_B T/I)^{\frac{1}{2}}$. As a result in the neighbourhood of the threshold, the dependence on the distance from the threshold is nonlinear.

Equation (83.4) was obtained in the Born approximation. This approximation is also used in the numerical calculations (see, e.g., [42]).

84. EFFECT OF THE ELECTROMAGNETIC FIELD FLUCTUATIONS ON THE KINETIC PROCESSES IN A PARTIALLY IONIZED PLASMA

Using the method considered in sections 79, 80, we may include in Eqs. (80.15), (80.16) also the processes due to the interaction of the charge particles with a fluctuating transverse electromagnetic field [37]. The additional terms in the collision integrals, due to the fluctuations of the electromagnetic field are expressed by similar formulae. For instance, the additional term to $[I_a(\boldsymbol{p}_a', t)]_1$ has the following form [37]:

$$\begin{aligned}\left[I_a(\boldsymbol{p}_a', t)\right]_1 = & \frac{e_a^2}{2(2\pi)^3\hbar} \int d\boldsymbol{p}_a''\, d\omega\, d\boldsymbol{k}\, \frac{(\boldsymbol{k}\times\boldsymbol{v}_a')^2}{k^2\omega^2}\, \delta(\hbar\boldsymbol{k} - \boldsymbol{p}_a' + \boldsymbol{p}_a'') \\ & \delta\left(\hbar\omega - \frac{p_a'^2}{2m_a} + \frac{p_a''^2}{2m_a}\right) \left\{\left(\delta E^\perp \cdot \delta E^\perp\right)_{\omega,\boldsymbol{k}} \left[f_a(\boldsymbol{p}_a'', t) - f_a(\boldsymbol{p}_a', t)\right]\right. \\ & \left. - \frac{8\pi\hbar \operatorname{Im} \varepsilon^\perp(\omega,\boldsymbol{k})\omega^4}{|\omega^2\varepsilon^\perp(\omega,\boldsymbol{k}) - c^2k^2|^2} \left[f_a(\boldsymbol{p}_a'', t) + f_a(\boldsymbol{p}_a', t)\right]\right\}.\end{aligned} \tag{84.1}$$

Here, $\left(\delta\boldsymbol{E}^\perp \cdot \delta\boldsymbol{E}^\perp\right)_{\omega,\boldsymbol{k}}$ is the spectral density of the transverse electric field fluctuation, $\varepsilon^\perp(\omega,\boldsymbol{k})$ is the corresponding dielectric constant. As in the Coulomb case, these functions can be represented as follows (see (80.7)):

$$\begin{aligned}\left(\delta\boldsymbol{E}^\perp \cdot \delta\boldsymbol{E}^\perp\right)_{\omega,\boldsymbol{k}} &= \left(\delta\boldsymbol{E}^\perp \cdot \delta\boldsymbol{E}^\perp\right)^{ff}_{\omega,\boldsymbol{k}} + (\cdots)^{fb} + (\cdots)^{bf} + (\cdots)^{bb}, \\ \varepsilon^\perp(\omega,\boldsymbol{k}) &= 1 + 4\pi(\alpha_{ff} + \alpha_{fb} + \alpha_{bf} + \alpha_{bb})\end{aligned} \tag{84.2}$$

The expressions of these functions were derived in reference [37].

In equilibrium, the spectral density of the transverse field fluctuations

is given by

$$\left(\delta \boldsymbol{E}^{\perp} \cdot \delta \boldsymbol{E}^{\perp}\right)_{\omega, \boldsymbol{k}} = \frac{16 \pi \hbar \omega^4 \operatorname{Im} \varepsilon^{\perp}(\omega, \boldsymbol{k})}{|\omega^2 \varepsilon^{\perp}(\omega, \boldsymbol{k}) - c^2 k^2|^2} \left(\frac{1}{2} + \frac{1}{\exp(\hbar\omega / k_B T) - 1}\right) . \tag{84.3}$$

For $\hbar \to 0$, this expression reduces to (39.42). The corresponding expression for the time-spectral density is

$$\left(\delta \boldsymbol{E}^{\perp} \cdot \delta \boldsymbol{E}^{\perp}\right)_{\omega} = \frac{4 \hbar \omega^3}{c^3} (\operatorname{Re} \varepsilon^{\perp})^{\frac{1}{2}} \left(\frac{1}{2} + \frac{1}{\exp(\hbar\omega / k_B T) - 1}\right)$$
$$\equiv \frac{2 \hbar \omega^3}{c^3} (\operatorname{Re} \varepsilon^{\perp})^{\frac{1}{2}} + 4 \pi^2 \rho_{\omega}^{\mathbf{T}} \tag{84.4}$$

where $\rho_{\omega}^{\mathbf{T}}$ is the temperature part of the Planck function.

Using (84.2), we may represent the collision integrals as sums of eight terms and introduce a classification of all the contributions, as in section 80.

The collision integrals take into account the interactions in the polarization approximation. Therefore, in addition to the usual processes of photoionization, photo-recombination, emission, absorption, etc., there also appear 'anomalous' (superluminous) effects [43]. Such are the Cherenkov effect, the anomalous Doppler effect, the anomalous bremmstrahlung, as well as the spontaneous and induced ionization of atoms with emission of a quantum, of the induced recombination with absorption of a quantum, etc.

For the purpose of illustration, we quote the expression of the coefficients of 'normal' photo-ionization, and photo-recombination (α_i and β_i) in the approximation of an infinite mass for the heavy particles [37]:

$$\beta_i = \frac{V}{2\hbar} (2\pi m_e k_B T)^{-\frac{3}{2}} \sum_m \int d\omega \, d\boldsymbol{p} \left| \boldsymbol{r}_{\boldsymbol{p}m}^{\perp} \right|^2 \delta\left(\hbar\omega - |E_m| - \frac{p^2}{2m_e}\right)$$
$$\cdot \left(\delta \boldsymbol{E}^{\perp} \cdot \delta \boldsymbol{E}^{\perp}\right)_{\omega} \exp(-p^2 / 2 m_e k_B T) . \tag{84.5}$$

Here $|\boldsymbol{r}_{\boldsymbol{p}m}^{\perp}|^2$ is the corresponding matrix element, $\left(\delta \boldsymbol{E}^{\perp} \cdot \delta \boldsymbol{E}^{\perp}\right)_{\omega}$ is the spectral function of the field, excluding the contribution of the ground-state oscillations. In equilibrium, this function reduces to $4\pi^2 \rho_{\omega}^{\mathbf{T}}$ (see (84.4)). The photo-ionization coefficient can be obtained from (84.5) by means of the substitution

$$\left(\delta \boldsymbol{E}^{\perp} \cdot \delta \boldsymbol{E}^{\perp}\right)_{\omega} \longrightarrow \left(\frac{m_e k_B T}{2\pi\hbar^2}\right)^{\frac{3}{2}} \frac{1}{Z} \exp\left(-\frac{\hbar\omega}{k_B T}\right) \left(\delta \boldsymbol{E}^{\perp} \cdot \delta \boldsymbol{E}^{\perp}\right)_{\omega} . \tag{84.6}$$

For the process of recombination in the ground state, Eq. (84.5) takes the form [37] :

$$\beta_i = \frac{2^8 e^{-4}}{3\hbar} \left(\frac{2\pi\hbar^2}{m_e k_B T}\right)^{\frac{3}{2}} a_0^3 \int_{I/\hbar}^{\infty} d\omega \exp\left(-\frac{\hbar\omega - I}{k_B T}\right) \left(\delta \boldsymbol{E}^{\perp} \cdot \delta \boldsymbol{E}^{\perp}\right)_{\omega} \tag{84.7}$$

The corresponding photoionization coefficient is obtained by means of (84.6) :

$$\alpha_i = \frac{2^8 e^{-4}}{3\hbar} a_0^3 \int_{I/\hbar}^{\infty} d\omega \left(\delta E^{\perp} \cdot \delta E^{\perp}\right)_{\omega} . \tag{84.8}$$

If the fluctuations of the field are in equilibrium, i.e., when

$$\left(\delta E^{\perp} \cdot \delta E^{\perp}\right)_{\omega} = 4\pi^2 \rho_{\omega}^{\mathrm{T}}$$

and $I \gg k_{\mathrm{B}}T$, we obtain from (84.8) [37] :

$$\alpha_i = \frac{2^{10} e^{-4}}{3} a_0^3 \frac{I^3 \left[\mathrm{Re}\, \varepsilon(I/\hbar)\right]^{\frac{1}{2}}}{\hbar^3 c^3} \frac{k_{\mathrm{B}} T}{\hbar} \exp\left(-I/k_{\mathrm{B}}T\right) . \tag{84.9}$$

This result differs only by a numerical factor from the result of the quasi-classical method (see, e.g., Eq. (6.96) in ref. [41]). More details about these problems are found in ref. [37] .

85. KINETIC THEORY OF FLUCTUATIONS IN CHEMICALLY REACTING GASES AND IN PARTIALLY IONIZED PLASMAS †

In deriving the kinetic equation of this chapter, we used the assumption of complete weaking of the correlations; therefore, the kinetic fluctuations do not appear in these equations. The latter can, however, be studied by analogy with the theory developed in Chapters 4 and 11.

As a starting point, we must use instead of (79.10) the equation for the smoothed density matrix. In contrast to (79.10), the latter equation is dissipative.

The kinetic equation, taking into account the long-range fluctuations is now:

$$\frac{\partial f_\alpha(\boldsymbol{P}', t)}{\partial t} = I_\alpha(\boldsymbol{P}', t) + \tilde{I}_\alpha(\boldsymbol{P}', t) . \tag{85.1}$$

The integral I_α is defined by (80.3), whereas

$$\tilde{I}_\alpha(\boldsymbol{P}', t) = -\frac{2}{\hbar} \sum_{\beta} \frac{V}{(2\pi\hbar)^3} \int d\boldsymbol{P}'' \,\mathrm{Im} \left\langle \delta\tilde{\rho}_{\alpha\beta}(\boldsymbol{P}', \boldsymbol{P}'', t)\; \delta\tilde{U}_{\alpha\beta}(\boldsymbol{P}', \boldsymbol{P}'', t)\right\rangle \tag{85.2}$$

is an additional term, taking into account the long-range fluctuations.

We recall that, in the kinetic theory of fluctuations in both gases and plasmas, the source of long-range fluctuations is a sum of two terms (see (22.26), (62.17)). One of them is due to the correction to the collision integral, $\tilde{I}_\alpha$, and vanishes when $\tilde{I}_\alpha = 0$. The second term is due to the atomic structure of the medium, and remains non-zero even in equilibrium, when both collision integrals I_α, $\tilde{I}_\alpha$, vanish.

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† This section was added by the author for the present English edition (1977).

The contribution due to the collision integral $\tilde{I}_\alpha$ is important when the state of the system is far from equilibrium. Here, we shall only discuss the case when $\tilde{I}_\alpha$ is zero. In that case we may assume that the matrix $\delta\tilde{\rho}_{\alpha\beta}(\boldsymbol{P}', \boldsymbol{P}'', t)$ is diagonal, i.e.:

$$\delta\tilde{\rho}_{\alpha\beta}(\boldsymbol{P}', \boldsymbol{P}'', t) = \delta_{\alpha\beta} \frac{(2\pi\hbar)^3}{V} \delta(\boldsymbol{P}'-\boldsymbol{P}'')\, \delta f_\alpha(\boldsymbol{P}', t) . \tag{85.3}$$

Here δf_α is the fluctuation in the distribution function f_α: $\delta f_\alpha = f_\alpha - \langle f_\alpha \rangle$.

In the present approximation the equation for the distribution function can be written as a Langevin equation

$$\frac{\partial f_\alpha}{\partial t} = I_\alpha(\boldsymbol{P}', t) + \frac{V}{N}\, y_\alpha(\boldsymbol{P}', t) . \tag{85.4}$$

The moments of the random source are given by

$$\langle y_\alpha(\boldsymbol{P}, t) \rangle = 0$$

$$\langle y_\alpha(\boldsymbol{P}', t)\, y_\beta(\boldsymbol{P}'', t) \rangle = A_{\alpha\beta}(\boldsymbol{P}', \boldsymbol{P}'', t)\ \delta(t-t') . \tag{85.5}$$

The intensity of the random source can be obtained by formulae similar to those of sections 22, 62.

In the local-equilibrium approximation we obtain from (85.1) the corresponding Langevin equations for the concentrations of electrons, ions and atoms:

$$\frac{dn_e}{dt} - \Big[\left(\alpha n_e n_{ei} - \beta n_e^2 n_i\right) + \left(\alpha_1 n_{ei}^2 - \beta_1 n_e n_i n_{ei}\right) + \left(\alpha_2 n_{ei}^2 - \beta_2 n_e^2 n_i^2\right)\Big] = y_e(t)$$

$$n_e = n_i , \quad n_e + n_{ei} = \frac{N}{V} . \tag{85.6}$$

The last term of Eq. (83.1) does not appear here because the coefficients $\alpha_3 = \beta_3$; therefore the term does not contribute to the concentration balance.

The moments of the Langevin sources are given by:

$$\langle y_e(t) \rangle = 0 , \quad \langle y_e(t)\, y_e(t') \rangle = A_{ee}(t)\ \delta(t-t') . \tag{85.7}$$

The expression for A_{ee} follows from the general expression $A_{\alpha\beta}(\boldsymbol{P}', \boldsymbol{P}'', t)$ given in (85.4). When the Maxwell-Boltzmann distribution is reached but not chemical equilibrium, one finds the following expression for A_{ee} [Belyi and Klimontovich]:

$$A_{ee}(t) = V^{-1} \Big\{\left(\alpha n_e n_{ei} + \beta n_e^2 n_i\right) + \left(\alpha_1 n_{ei}^2 + \beta_1 n_{ei} n_e n_i\right) + 2\left(\alpha_2 n_{ei}^2 + \beta_2 n_e^2 n_i^2\right)\Big\} . \tag{85.8}$$

Thus, the intensity of the random source involves three terms, each corresponding to one of the pair (direct and inverse) of processes defining the change in the concentrations.

In a state of complete equilibrium (when the Saha law is valid) the first and second term in each pair in (85.8) become equal.

For small departures from equilibrium Eq. (85.6) reduces in the linear approximation for δn_e to

$$\frac{d}{dt}\delta n_e + \lambda \delta n_e = y_e , \qquad \lambda = \alpha \left(2\frac{N}{V} - n_e^0 \right) ,$$

$$\delta n_i = \delta n_e , \qquad \delta n_{ei} = - \delta n_e . \tag{85.9}$$

For simplicity we retained here only the term describing the ionization and the recombination. We obtain from this equation an expression for the mean square of the total number of electrons $\delta N_e = V \delta n_e$:

$$\langle (\delta N_e)^2 \rangle = N_e \frac{N - N_e}{2N - N_e} . \tag{85.10}$$

Clearly, in both limiting cases of a fully ionized plasma and of a vanishing degree of ionization this expression vanishes.

Using the relations between the fluctuations δn_e, δn_i and δn_{ei} one easily finds expressions for $\langle (\delta N_i)^2 \rangle$ and $\langle (\delta N_{ei})^2 \rangle$.

The concentration fluctuations can, of course, be calculated also by other methods (see, for instance, the work of van Kampen, Brenig, Horsthemke, Grossmann, etc.). The method sketched here can be used even in states far from equilibrium when the contribution of the long-range fluctuations becomes important.

The present chapter must be viewed as a short introduction to the theory of partially ionized gases and of chemically reacting gases. In this field there are still many unsolved problems. One of them is the construction of the kinetic theory of nonideal, chemically reacting systems.

References

NOTES: The symbol (R) denotes an original reference in Russian language

The symbol (E) denotes an original reference in English language

The symbol (ET) denotes an English translation of an original reference in another language. The translator has provided these, as far as possible.

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